

An evolution model of parametric surface deformation using finite elements based on B-splines.

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In this paper we introduce a dynamic evolution model in order to deform parametric surfaces. In order to do it, we present the associate variational formulation to the problem of minimize an energy functional. Its numerical resolution is developed using a finite element method based on B-splines. We compute the spatial discretization where the finite elements are defined and we show that a reduced number of control points are deformed instead of all the surface points.

1 INTRODUCTION

The deformation models include a large number of applications, and they have been used in fields as the edge detection, computer animation, geometric modelling, and so on. In this work, a deformation model will be introduced that uses B-splines as finite elements. This theory was introduced by Höllig in (Höllig 2003). In fact, we have used a variational formulation similar to the used one in (Cohen 1992) changing, among other things, the selected finite elements.

The most used finite elements for surfaces are the triangles, squares, among others; with them, it is easy to make a mosaic that it fills all space, but this technique needs long computation time since we must use some big data structures to solve our problem. This data structure are bound by the quantity of surface points that we must to take to obtain a good approximation of the surface.

In this work, we have applied the finite elements method based on B-splines to solve numerically a partial differential equation problem. The advantage to use B-splines finite elements is that it combines the computational advantage of B-splines and standard mesh-based elements. Thus, we will obtain a data structure smaller than the obtained one using the usual finite elements.

This work is organized as follows. In section 2, we define the uniform B-splines used to introduce finite elements. First, we define one dimensional B-splines and we display a recurrence relation to evaluate the numerical value in a point of the B-spline. Moreover we present a recurrence formula in order to compute the B-splines derivatives. Next, we define the multivariate B-splines. To finish this section, we define a B-spline parametric surface. Section 3 is devoted to introduce the model of surface deformation to which we will apply the finite elements methods using B-splines. The numerical resolution is focused in section 4. First, we raised the variational formulation to follow with its numerical resolution, introducing the fi-

nite element B-splines. Next, we compute the spatial and temporal discretization where the finite elements are defined. We will show that a subset of the control points are deformed instead of all the surface points. The model evolution will be also introduced. In the next section, several numerical computer deformations are displayed using the evolution model with different forces. Finally, some conclusions are exposed.

2 B-SPLINES

The B-splines are piecewise polynomial functions. It has been verified, with others approximations functions technics (Piegl 1997) that the polynomials provide a good local approximation for smooth functions. However, if we use large intervals, accuracy of the approximation can be very low, the exactitude of the approach could be very low and local changes have global influence. Therefore, it is natural to use piecewise polynomials, defined on a fine partition of the function domain. For this reason, we will use piecewise polynomials approximation, and of between all of them, we choose the B-splines.

2.1 B-SPLINES FUNCTIONS

Uniform B-splines can be defined in several ways (de Boor 1978) (Farin 1997) (Piegl 1997) (Höllig 2003). In this work we have taken the definition given by Höllig in (Höllig 2003), which we describe next.

Definition 2.1 (Höllig 2003) *An uniform B-spline of degree n , b^n , is defined by*

$$b^n(x) = \int_{x-1}^x b^{n-1}(t) dt$$

starting with $b^0(x) = \begin{cases} 1, & x \in [0, 1), \\ 0, & \text{otherwise.} \end{cases}$

The previous definition is not adapted for numerical evaluations. In order to be able to evaluate the B-splines in a simple form and fast computationally, we can use the recurrence equation. This equation was given by De Boor (de Boor 1978) and Cox (Cox 1972), and it is a linear combination of smaller degree B-splines.

$$b^n(x) = \frac{x}{n}b^{n-1}(x) + \frac{(n+1-x)}{n}b^{n-1}(x-1) \quad (1)$$

In order to construct the finite elements bases, we will use a scaled and translated uniform B-spline. They are defined by transforming the standart uniform B-spline, b^n , to the grid $h\mathbb{Z} = \{\dots, -2h, h, 0, h, 2h, \dots\}$, where h is the scaled step.

Definition 2.2 *The transformation for $h > 0$ and $k \in \mathbb{Z}$ is $b_{k,h}^n(x) = b^n(\frac{x}{h} - k)$. The support of this function is $[k, k + n + 1)h$*

In order to make a variational formulation of a differential equation problem, we will need the derivatives of the finite elements. From the definition 2.1 we obtain that the first order derivative of degree n B-spline is given by

$$\frac{d}{dx}b^n = b^{n-1}(x) - b^{n-1}(x-1)$$

with $b^n(0) = 0$ (Höllig 2003). If we apply the transformation given in definition 2.2, the first order derivative of the transformed B-spline is given by

$$\frac{d}{dx}b_{k,h}^n = h^{-1}(b_{k,h}^{n-1}(x) - b_{k+1,h}^{n-1}(x)) \quad (2)$$

(Höllig 2003) We also need the derivatives of any order. These ones are given by a linear combination of lower degree B-splines. The differentiation formula can be expressed in a compact form as follows.

Theorem 2.1 *The m^{th} derivative of a degree n transformed B-spline following the definition 2.2 is given by the recurrence relation*

$$\frac{d^m}{dx^m}b_{k,h}^n(x) = h^{-m} \sum_{i=0}^m (-1)^i \binom{m}{i} B_{k+i,h}^{n-m}(x) \quad (3)$$

Obviously, this equation has sense if $m \leq n$ since in others cases the derivative is 0.

2.2 MULTIVARIATE B-SPLINES

There is no unique generalization of one dimensional B-splines. These generalizations differs in the underlying partition for the polynomial segments (de Boor, Höllig, and Riemenschneider 1993) (Piegl 1997), (Höllig 2003), .

A possibility is to form products of one dimensional B-splines, as described in the following construction. The N -variate B-spline of degree $\mathbf{n} = (n_1, \dots, n_N)$, of index $\mathbf{k} = (k_1, \dots, k_N)$ and the space discretization $\mathbf{h} = (h_1, \dots, h_N)$ is defined as

$$B_{\mathbf{k},\mathbf{h}}^{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^N b_{k_i,h_i}^{n_i}(x_i). \quad (4)$$

The support of this function is $\prod_{i=1}^N [k_i, k_i + n_i + 1)h_i$. Applying basic properties of differential calculus and applying theorem 2.1 we can obtain a compact expression for any partial derivative of multivariate B-spline. We note that using theorem 2.1, we can evaluate the derivatives and they are obtained with a smaller computational cost, since less recurrences are applied.

In the next figure we show the graph of several bidimensional B-splines.

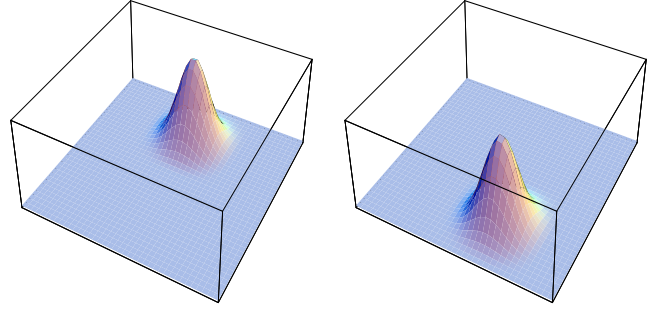


Figure 1: Bicubic B-splines with scaled step $h = 1/6$. Left: using a translation $k = (4, 9)$. Right: using a translation $k = (9, 0)$.

2.3 PARAMETRIC SURFACES WITH B-SPLINES

A parametric surface is defined as $S : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $(u, v) \mapsto S(u, v) = (x(u, v), y(u, v), z(u, v))$ with the necessary degree of differentiability (do Carmo 1976), where Ω is a bounded bidimensional subset. This surface will be B-spline if we can put it as a linear combination of bidimensional B-splines. That is,

$$S(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^2} P_{\mathbf{k}} B_{\mathbf{k},\mathbf{h}}^{\mathbf{n}}(\mathbf{x}) \quad (5)$$

where $\mathbf{n} \in \mathbb{N}^2$, and $\mathbf{h} \in \mathbb{R}^2$ with positive coordinates. The coefficients $P_{\mathbf{k}} \in \mathbb{R}^3$ are called *control points* and they are the elements that determine the B-spline surface.

In order to be able to work with finite elements, we will need bases with a finite number of elements. The parametric surfaces we will use must have a bounded domain. Consequently, all $P_{\mathbf{k}}$ will be zero except a finite number of them. In order to find this set of control points, we must find the relevant B-splines. These ones fulfill $Sup(B_{\mathbf{k},\mathbf{h}}^{\mathbf{n}}) \cap \Omega \neq \emptyset$.

The relevant B-splines of our surface are determined by the spatial discretization, since the B-splines support depends on them. This problem will be addressed in the section of numerical resolution (section 4).

3 DEFORMATION MODEL. MINIMAL SURFACES

The deformation model is based on an associated energy to one surface, that it checks the shape of it.

The energy function is a non convex function with a local minimum. The goal is to achieve this minimum using an evolution model. This minimum depends on the initial surface and the used evolution model.

The associated energy functional, $E : \Phi(S) \rightarrow \mathbb{R}$, $S \mapsto$

$E(S)$, is defined as

$$E(S) = \int_{\Omega} \omega_{10} \left| \frac{\partial S}{\partial u} \right|^2 + \omega_{01} \left| \frac{\partial S}{\partial v} \right|^2 + \omega_{11} \left| \frac{\partial S}{\partial u \partial v} \right|^2 + \omega_{20} \left| \frac{\partial^2 S}{\partial u^2} \right|^2 + \omega_{02} \left| \frac{\partial^2 S}{\partial v^2} \right|^2 + \mathcal{P}(S(u, v)) dudv$$

(Terzopoulos 1986), (Cohen 1992), (Montagnat, Delingette, and Ayache 2001), where \mathcal{P} is a potential of the forces that works on the surface. Using the equations of Euler-Lagrange, it can be proved (Cohen 1992) that an energy local minimum must be satisfied:

$$-\omega_{10} \frac{\partial^2 S}{\partial u^2} - \omega_{01} \frac{\partial^2 S}{\partial v^2} + 2\omega_{11} \frac{\partial^4 S}{\partial u^2 \partial v^2} + \omega_{20} \frac{\partial^4 S}{\partial u^4} + \omega_{02} \frac{\partial^4 S}{\partial v^4} = -\nabla \mathcal{P}(S(u, v)) + \text{boundary conditions} \quad (6)$$

The surface domain is $\Omega = [0, 1]^2$ and we take as boundary conditions $S(u, 0) = (u, 0, 0)$, $S(u, 1) = (u, 1, 0)$, $S(0, v) = (0, v, 0)$, $S(1, v) = (1, v, 0)$.

4 NUMERICAL RESOLUTION

4.1 VARIATIONAL FORMULATION

With the purpose of establishing the variational formulation of the boundary value problem done by (6), we recall the definition of a Sobolev Space of order two $H^2(\Omega)$,

$$H^2(\Omega) = \{S \in L^2(\Omega) : \frac{\partial^{(\alpha_1 + \alpha_2)} S}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \in L^2(\Omega), 0 \leq \alpha_1 + \alpha_2 \leq 2, \alpha_1, \alpha_2 \in \mathbb{N}\}$$

We will consider the set of functions $(H^2(\Omega))^3$ satisfying the previous boundary conditions. We will denote this set by \mathcal{H} .

The weak formulation of the equation (6) is:

$$\int_{\Omega} \left(\omega_{10} \frac{\partial S}{\partial u} \frac{\partial T}{\partial u} + \omega_{01} \frac{\partial S}{\partial v} \frac{\partial T}{\partial v} + 2\omega_{11} \frac{\partial^2 S}{\partial u \partial v} \frac{\partial^2 T}{\partial u \partial v} + \omega_{20} \frac{\partial^2 S}{\partial u^2} \frac{\partial^2 T}{\partial u^2} + \omega_{02} \frac{\partial^2 S}{\partial v^2} \frac{\partial^2 T}{\partial v^2} \right) dudv = - \int_{\Omega} \nabla \mathcal{P}(S) T dudv \quad (7)$$

where the functions S, T belongs to \mathcal{H} and u, v are the spatial variables.

It can be proved (Cohen 1992) (Raviart 1992) that to solve equation (6) is equivalent to find an element $S \in \mathcal{H}$, such that $a(S, T) = L(T)$ for all $T \in \mathcal{H}$. Where $a(\cdot, \cdot)$ is a

bilinear form defined as

$$a(S, T) = \int_{\Omega} \left(\omega_{10} \frac{\partial S}{\partial u} \frac{\partial T}{\partial u} + \omega_{01} \frac{\partial S}{\partial v} \frac{\partial T}{\partial v} + 2\omega_{11} \frac{\partial^2 S}{\partial u \partial v} \frac{\partial^2 T}{\partial u \partial v} + \omega_{20} \frac{\partial^2 S}{\partial u^2} \frac{\partial^2 T}{\partial u^2} + \omega_{02} \frac{\partial^2 S}{\partial v^2} \frac{\partial^2 T}{\partial v^2} \right) dudv \quad (8)$$

and $L(\cdot)$ is the following linear form

$$L(T) = - \int_{\Omega} \nabla \mathcal{P}(S) T dudv$$

4.2 DISCRETIZATION

We want to find a function $S \in \mathcal{H}$ such that

$$a(S, T) = L(T) \quad \forall T \in \mathcal{H} \quad (9)$$

In order to do this, the surface domain will be discretized. But, first of all, we have to find a set of functions of finite dimension. The B-splines defined in section 2 will be the finite elements that we will use as the base of our function set. The problem is to find the relevant B-splines. That is, the B-splines satisfying $Sup(B_{\mathbf{k}, \mathbf{h}}^n) \cap \Omega \neq \emptyset$, and from this, the set of index \mathbf{k} of the B-splines that satisfies the boundary conditions. Therefore, we want to evolve $N_1 \times N_2$ control points of the B-spline surface S . To do it, we need $N_1 \times N_2$ bidimensional B-splines such that its support will be in Ω .

The solution $S \in \mathcal{H}$ is a B-spline surface of degree $n = (n_x, n_y)$. The surface domain is discretized by $h_1 \mathbb{Z} \times h_2 \mathbb{Z}$ where $h_1 = \frac{1}{N_1 + n_x - 1}$ and $h_2 = \frac{1}{N_2 + n_y - 1}$. This spatial discretization will fix the control points that are not zero. The index $\mathbf{k} = (k_1, k_2)$ belongs to the set $\{-n_x, \dots, N_1 + n_x - 1\} \times \{-n_y, \dots, N_2 + n_y - 1\}$. So, the B-spline surface will come determined by the relevant B-splines, and they are specified in the following equation

$$S(u, v) = \sum_{k_1 = -n_x}^{N_1 + n_x - 1} \sum_{k_2 = -n_y}^{N_2 + n_y - 1} P_{(k_1, k_2)} B_{(k_1, k_2)\mathbf{h}}^n(u, v) \quad (10)$$

In addition, deforming only the corresponding $N_1 \times N_2$ control points of the B-spline surface (10), we made sure that the boundary conditions are satisfied.

The set of finite elements of finite dimension that determines our B-spline bases is given by $V_{\mathbf{h}}^n = \{ (B_{\mathbf{k}, \mathbf{h}}^n(u, v), 0, 0) : \mathbf{k} \in \mathcal{I} \} \cup \{ (0, B_{\mathbf{k}, \mathbf{h}}^n(u, v), 0) : \mathbf{k} \in \mathcal{I} \} \cup \{ (0, 0, B_{\mathbf{k}, \mathbf{h}}^n(u, v)) : \mathbf{k} \in \mathcal{I} \}$ where $\mathcal{I} = \{0, \dots, N_1 - 1\} \times \{0, \dots, N_2 - 1\}$.

Thus, taking into account the boundary conditions, the control points $P_{\mathbf{k}}$ associated to B-splines belonging to the set $V_{\mathbf{h}}^n$ are the unique ones that are computed using the equations (11) and (12) (see below).

Using equations (9) and (10) we can obtain three linear systems, one for each coordinate:

$$AP_i = L_i, \quad i = 1, 2, 3, \quad (11)$$

where A is a square matrix and their elements are:

$$a((B_{k,h}^n, 0, 0), (B_{j,h}^n, 0, 0))_{(k,j) \in \mathcal{I} \times \mathcal{I}},$$

P_i is a vector of component i of each control point and L_i is a vector with components $L_1 = L((B_{k,h}^n, 0, 0))_{k \in \mathcal{I}}$, $L_2 = L((0, B_{k,h}^n, 0))_{k \in \mathcal{I}}$, $L_3 = L((0, 0, B_{k,h}^n))_{k \in \mathcal{I}}$.

The static problem has been introduced. Next, we will construct the evolution model. The classical dynamical model of evolution has been applied (Cohen 1992), (Qin 1997), (Montagnat, Delingette, and Ayache 2001), (González, Mascaró, Mir, Palmer, and Perales 2001), (Mascaró 2002) (Mascaró, Mir, and Perales 2002):

$$M \frac{d^2 P_i}{dt^2} + C \frac{dP_i}{dt} + AP_i = L_i, \quad i = 1, 2, 3. \quad (12)$$

where M and C are the mass and damping matrices respectively and are diagonal matrices. The dynamic system (12) has been discretized in time by the finite difference scheme using central differences, obtaining an explicit numerical scheme.

When we applied the dynamic model, the surface depends on time. So, we have $S(u, v, t)$. Nevertheless, this dependency only affects to the control points which is an advantage since in each iteration we do not have to calculate all the surface. Therefore, we only must calculate the new control points. The used numerical scheme in the dynamic model depends on two previous iterations $P^{t-\Delta t}$ and P^t , but as it is not a physical method, we have taken the same P^0 and P^1 , where $P^0 = S(u, v, 0)$.

5 EXAMPLES

This section shows several examples of deformations obtained applying our dynamical model. All the experiments displayed in this section has been made using $\omega_{10} = \omega_{01} = 0.1$ and $\omega_{11} = \omega_{20} = \omega_{02} = 0.01$ and taking a temporal step $t = 0.1$.

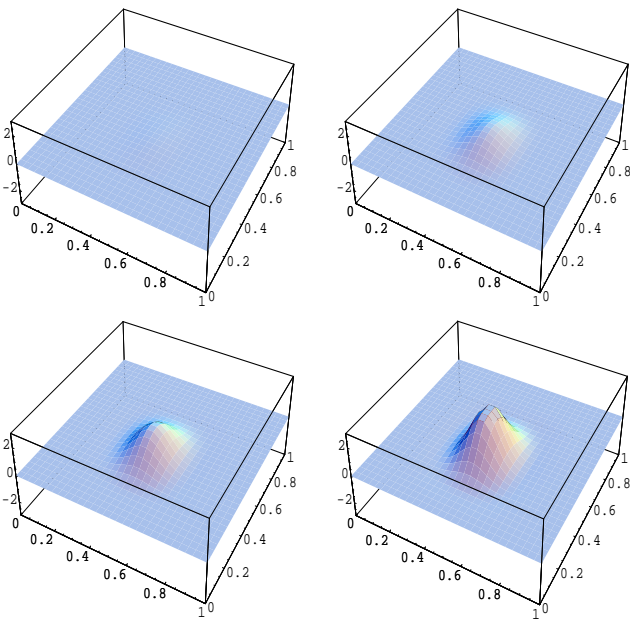


Figure 2: Dynamic simulation of a plane deformation using a positive constant force with direction $(0, 0, 1)$.

In Figure 2, we show several iterations obtained using the dynamic model with bicubic B-splines, a positive force in the direction $(0, 0, 1)$ and $N_1 \times N_2 = 25$. The force is applied in an unique point of the surface.

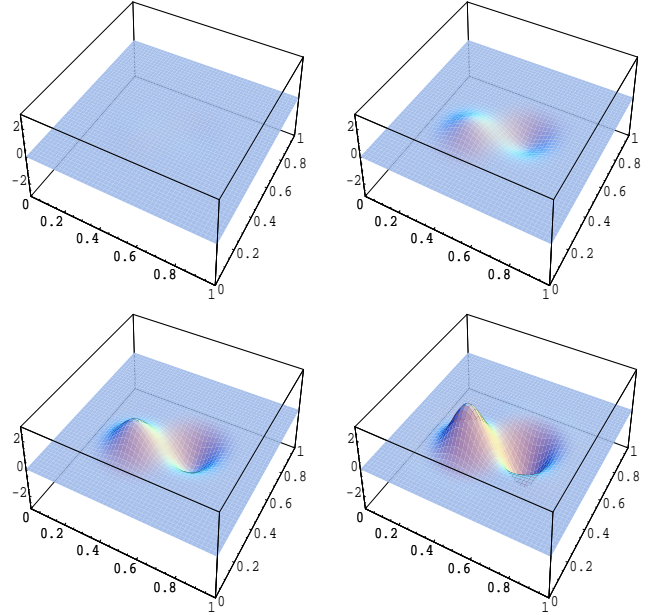


Figure 3: Dynamic simulation of a plane deformation using two constant forces simultaneously in opposite sense.

Figure 3 shows the deformation obtained using bicubic B-splines, two forces simultaneously in opposite sense in the direction $(0, 0, 1)$ and $N_1 \times N_2 = 25$. The force is applied in two nearly points of the surface.

Not only we can apply forces in vertical directions. Also, we can apply forces in other directions as we can see in Figure 4, where we have applied over all the surface domain an oblique force in the direction $(1, 1, 4)$, with module 106, taking bicubic B-splines and $N_1 \times N_2 = 49$.

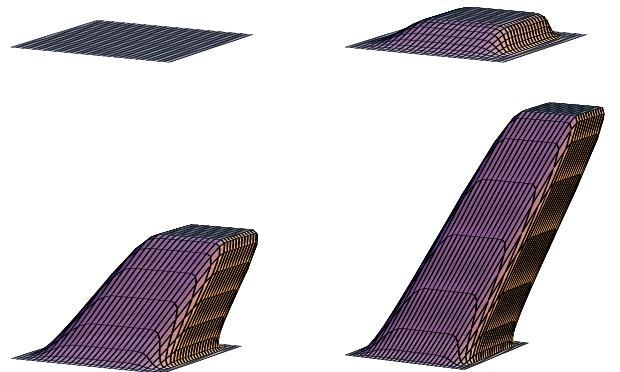


Figure 4: Dynamic simulation of a plane deformation using a force in the direction $(1, 1, 4)$

As we can observe in equation 8, the bilinear form has

five parameters. They represent the resistance to the deformation. In Figure 5, we can see the same iteration (iteration 6) of the experiment displayed in Figure 4 changing the bilinear form parameters. From top to bottom and from left to right these parameters are: $\omega_{10} = \omega_{01} = 0.1$ and $\omega_{11} = \omega_{20} = \omega_{02} = 0.01$ (see figure 4); $\omega_{10} = \omega_{01} = 0.1$, $\omega_{11} = 1$ and $\omega_{20} = \omega_{02} = 0.01$; $\omega_{10} = \omega_{01} = 10$, $\omega_{11} = 1$ and $\omega_{20} = \omega_{02} = 0.01$ and $\omega_{10} = \omega_{01} = 100$, $\omega_{11} = 1$ and $\omega_{20} = \omega_{02} = 0.01$, respectively.

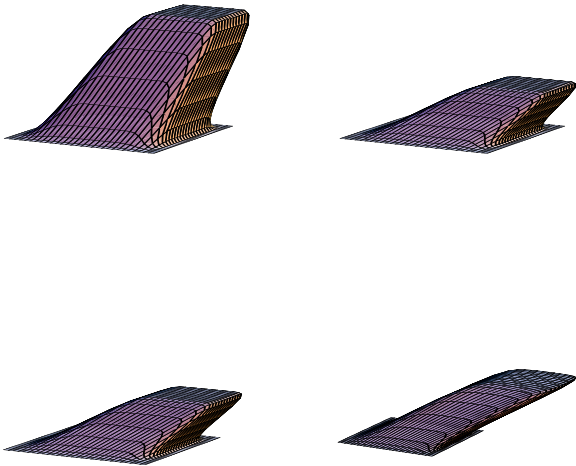


Figure 5: Several iterations of a plane deformation using the force $(1, 1, 4)$ and different values of the bilinear form parameters. For details see the text.

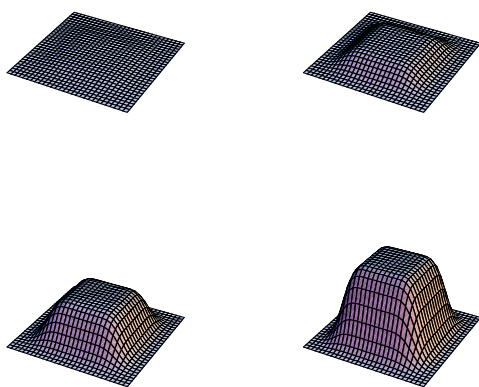


Figure 6: Several iterations of a plane deformation using the vertical force $(0, 0, 200)$ over all the surface.

Figure 6 shows several iterations obtained using the vertical force $(0, 0, 200)$ over all the surface, using bicubic B-splines and $N_1 \times N_2 = 36$. We can compare this deformation with the obtained one in Figure 2.

The next figures, Figure 7 and Figure 8, display experiments using sinusoidal forces. In Fig-

ure 6 we use the force given by $\nabla\mathcal{P}(u, v) = (50 \sin v\pi, 50 \sin u\pi, 200 \cos u\pi \sin v\pi)$, B-splines of degree $\mathbf{n} = (4, 4)$ and $N_1 \times N_2 = 36$. In Figure 7 we take the force defined by $\nabla\mathcal{P}(u, v) = (-50 \sin u\pi, 0, 200 \cos v\pi \sin u\pi)$, taking bicubic B-splines and $N_1 \times N_2 = 36$.

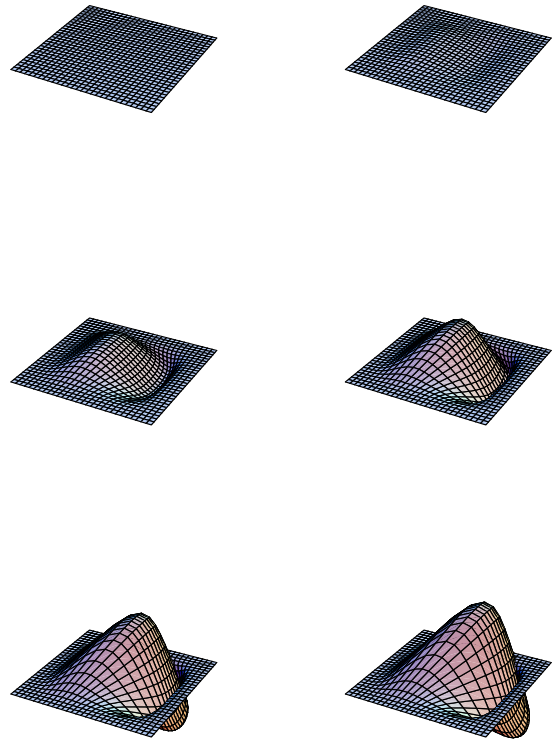


Figure 7: Dynamic simulation of a plane deformation using a sinusoidal force. Details in the text.

6 CONCLUSIONS AND FUTURE WORK

As we can see with the examples, we have shown the viability to make deformations of parametric surfaces using the variational formulation of the model, and solving it numerically using finite elements based on B-splines. In addition, as a certain number of control points only evolves, a very efficient scheme is obtained. In the context of future work we would like to highlight the following: a study of the stability and complexity of the model and a complete study of the influence of each parameter in the deformation obtained by the model. That is, the influence of them in the resistance to length deformation, to shear deformation and to bend deformation, as well as the use of other energy functional, among others. At the present, we are working on the implementation of this model using C++ and Coin3D, a 3D modeling toolkit which simplifies visualization and scene composition tasks.

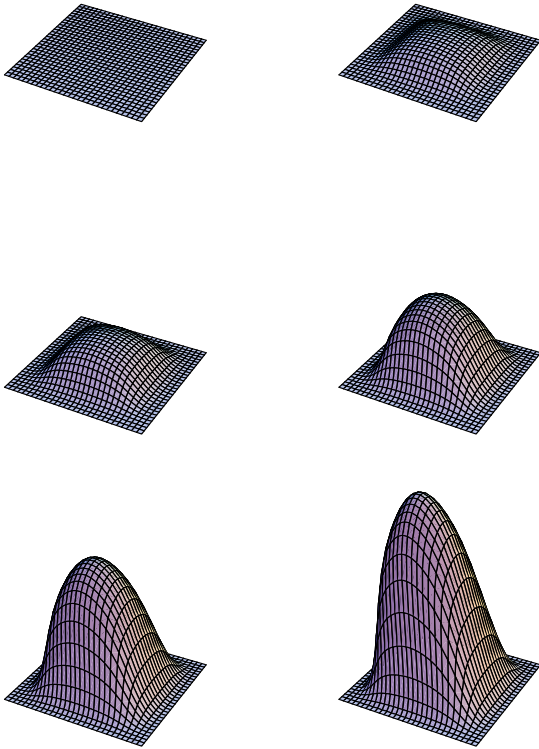


Figure 8: Deformations obtained using other sinusoidal force different from the applied one in the figure 7. Details in the text.

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