Opening and Closing Operators in Fuzzy Morphology Using Conjunctive Uninorms

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Abstract

In this paper, a fuzzy mathematical morphology is defined using conjunctive uninorms. Implementation results for two special classes of representable and idempotent conjunctive uninorms are presented, proving that these classes become specially appropriate for edge detection. Moreover, open and closed fuzzy objects are defined using the mentioned kinds of uninorms which leads us to prove, for the case of representable ones, the generalized idempotence for fuzzy opening and closing.

Keywords: Fuzzy morphology, erosion, dilation, opening, closing, uninorms, implicators.

1 Introduction

The fuzzy mathematical morphology is an alternative extension of binary morphology to gray-scale morphology using concepts and techniques from fuzzy set theory (see for example [1],[2],[7],[8]). The basic tools of mathematical morphology are the morphological operations on an image A, which are defined relatively to a fuzzy structuring element B, the size and shape of which can be chosen by the morphologist in order to analize the structure of A. Several researchers have introduced alternative morphological operations, a detailed account can be found in [11], and references therein.

Following the general framework for fuzzy mathematical morphology constructed by De Baets in [2] where he uses "conjunctors" and "implicators" in order to define the basic fuzzy morphological operations, we investigate in [14] which "conjuctive uninorms" (as a particular case of conjunctors) need to be chosen in order to preserve the algebraic and morphological properties needed to obtain a "good" mathematical morphology.

Our goal is to study, following the framework presented in [14], algebraic and characterization properties of "fuzzy closing" and "fuzzy opening", closed and open fuzzy objects, when we use conjunctive uninorms. Similar properties are obtained to those described by Be Baets in [15] for some particular conjunctors and, in a most general context by Bodenhofer in [16]. The paper is organized as follows. In Sections 2 and 3 we review the basic definitions and properties of fuzzy logical operators and those of fuzzy morphological operators, respectively. In Section 4 we discuss the properties of open and closed fuzzy objects and its representations. In particular, we prove that when we take a "conjunctive representable uninorm" we obtain the so-called "generalized idempotence law" for the fuzzy closing and fuzzy opening (see [15]). Some proofs are omitted to enlarge the last section, where we display some comparative experimental results using several conjunctive uninorms.

2 Fuzzy logical operators

Let us recall the fuzzy logical operators that we will use throughout the paper. More details on these operators can be found for instance in [12].

Definition 1. *A decreasing and involutive unary operator $\mathcal{N}$ on $[0,1]$ with $\mathcal{N}(0)=1$ and $\mathcal{N}(1)=0$ is called a strong negation.*

Definition 2. *An increasing binary operator $\mathcal{C}$ on $[0,1]$ is called a conjunctor if it satisfies*

$$\mathcal{C}(0,1)=\mathcal{C}(1,0)=0 \quad \textit{and} \quad \mathcal{C}(1,1)=1.$$

Definition 3. *A binary operator $\mathcal{I}$ on $[0,1]$ is called an implicator if it is decreasing with the first partial map, increasing with the second one and it satisfies*

$$\mathcal{I}(0,0)=\mathcal{I}(1,1)=1 \quad \textit{and} \quad \mathcal{I}(1,0)=0.$$

One can construct conjunctors and implicators from each other. On one hand, given an implicator $\mathcal{I}$ and a strong negation $\mathcal{N}$ the binary operator defined by

$$\mathcal{C}_{\mathcal{I},\mathcal{N}}(a,b)=\mathcal{N}(\mathcal{I}(a,\mathcal{N}(b)))$$

is a conjunctor. On the other hand, given a conjunctor $\mathcal{C}$ and a strong negation $\mathcal{N}$ the binary operator defined by

$$\mathcal{I}_{\mathcal{C},\mathcal{N}}(a,b)=\mathcal{N}(\mathcal{C}(a,\mathcal{N}(b)))$$

is an implicator. Another way to construct implicators from conjunctors is by residuation. Given a conjunctor $\mathcal{C}$ the binary operator

$$\mathcal{I}_{\mathcal{C}}(a,b)=\sup\{c\in[0,1] \mid \mathcal{C}(a,c)\leq b\}$$

is an implicator called the residual implicator of $\mathcal{C}$.

A special kind of conjunctors is given by the well known t-norms. In fact, fuzzy morphological operators are usually constructed from t-norms and, a special kind of them, the nilpotent ones, has been proved to be the most useful in this framework (see for instance [11]). However, a generalization of t-norms has appeared and has been studied in [10]:

Definition 4. *A uninorm is a two-place function $U:[0,1]\times[0,1]\longrightarrow[0,1]$ which is associative, commutative, increasing in each place and such that there exists some element $e\in[0,1]$, called the neutral element, such that $U(e,x)=x$ for all $x\in[0,1]$.*

It is clear that the function U becomes a t-norm when $e=1$ and a t-conorm when $e=0$. For any uninorm we have $U(0,1)\in\{0,1\}$ and a uninorm U is said conjunctive when $U(1,0)=0$ and disjunctive when $U(1,0)=1$. Moreover, a uninorm U is said to be idempotent whenever $U(x,x)=x$ for all $x\in[0,1]$.

This kind of operators results specially interesting because of their behavior: like a t-norm in $[0,e]^2$ and like a t-conorm in $[e,1]^2$. Moreover, note that conjunctive uninorms are particular cases of conjunctors and consequently they can be used in fuzzy mathematical morphology.

There are three known classes of conjunctive uninorms ([3]): uninorms in $\mathcal{U}_{\min}$, representable uninorms and idempotent uninorms. The first two classes have been already used in fuzzy morphology in [9]. Since left-continuity is essential in order to have "good" properties, and there is no left-continuous uninorms in the class $\mathcal{U}_{\min}$, we will only use here representable and idempotent, conjunctive uninorms. Of course, a fuzzy morphology can be done using also uninorms in $\mathcal{U}_{\min}$, but all properties stated and proved in this paper where left-continuity is required can fail, for this kind of logical operators. Let us recall here the definitions and characterizations of representable and idempotent uninorms, but more details of these classes can be found in [10] and [4] respectively.

Definition 5. *Let $e\in(0,1)$ and let $h:[0,1]\to[-\infty,+\infty]$ be a strictly increasing, continuous function with $h(0)=-\infty$, $h(e)=0$ and $h(1)=+\infty$. The binary operator $\mathcal{U}$ defined by*

$$\mathcal{U}(a,b)=h^{-1}(h(a)+h(b))$$

for all $(a,b)\in[0,1]^2\setminus\{(0,1),(1,0)\}$ and $\mathcal{U}(0,1)=\mathcal{U}(1,0)=0$ is a conjunctive uninorm with neutral element e. This kind of *uninorms are usually called representable conjunctive uninorms.*

Theorem 2.1. *A uninorm $\mathcal{U}$ with neutral element $e \in (0,1)$ is representable if and only if it is strictly increasing and continuous on $(0,1)^2$ and there is a strong negation $\mathcal{N}$ with $\mathcal{N}(e) = e$, such that for any $(a,b) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$*

$$\mathcal{U}(a,b) = \mathcal{N}(\mathcal{U}(\mathcal{N}(a),\mathcal{N}(b))).$$

Theorem 2.2. *Let $\mathcal{U}$ be a representable uninorm with additive generator h, then its residual implicator $I_{\mathcal{U}}$ is given by $I_{\mathcal{U}}(x,y) = h^{-1}(h(y) - h(x))$ if $(x,y) \in [0,1]^2 \setminus \{(0,0),(1,1)\}$ and $I_{\mathcal{U}}(0,0) = I_{\mathcal{U}}(1,1) = 1$.*

Between the idempotent uninorms, we will only use in this paper left-continuous, conjunctive, idempotent uninorms. However, note again that any other kind of conjunctive, idempotent uninorms (see [13]) can also be used in the same way.

Theorem 2.3. *A binary operator $\mathcal{U}$ is a left-continuous idempotent uninorm with neutral element $e \in (0,1)$ if and only if there exists a decreasing function $g : [0,1] \to [0,1]$ with fix point e, satisfying $g^2(x) \geq x$ for all $x \leq g(0)$ and $g(x) = 0$ for all $x > g(0)$ such that, for all $x,y \in [0,1]$, $\mathcal{U}$ is given by*

$$\mathcal{U}(x,y) = \begin{cases} \min(x,y) & \text{if } y \leq g(x) \text{ and } x \leq g(0) \\ \max(x,y) & \text{elsewhere.} \end{cases}$$

Note that given any strong negation $\mathcal{N}$ we obtain a left-continuous idempotent uninorm just taking $g = \mathcal{N}$, that we will denote by $\mathcal{U}^{\mathcal{N}}$. On the other hand, the residual implicator of an idempotent uninorm is given by

Theorem 2.4. *Let $\mathcal{U}$ be any idempotent uninorm with $g(0) = 1$. The residual implicator $I_{\mathcal{U}}$ is given by:*

$$I_{\mathcal{U}}(x,y) = \begin{cases} \min(g(x),y) & \text{if } \ y < x \\ \max(g(x),y) & \text{if } \ y \geq x. \end{cases}$$

Proposition 2.5. *(See [6]) Let $\mathcal{U}$ be a conjunctive uninorm and $I_{\mathcal{U}}$ its residual implicator.*

- *The second partial map of $I_{\mathcal{U}}$ is right-continuous and for all $x,y \in [0,1]$, $y \leq I_{\mathcal{U}}(x,\mathcal{U}(x,y))$.*
- *If $\mathcal{U}$ is left-continuous then so is the first partial map of $I_{\mathcal{U}}$, $I_{\mathcal{U}}$ satisfies the exchange principle:*

  $$I_{\mathcal{U}}(x, I_{\mathcal{U}}(y,z)) = I_{\mathcal{U}}(y, I_{\mathcal{U}}(x,z)),$$

  and also the following properties: $\mathcal{U}(x, I_{\mathcal{U}}(x,y)) \leq y$ and $I_{\mathcal{U}}(\mathcal{U}(x,y),z) = I_{\mathcal{U}}(x, I_{\mathcal{U}}(y,z))$ for all $x,y,z \in [0,1]$.

3 Fuzzy morphological operators

From the definition of classical erosion and dilation ([11]) it is clear that the intersection and inclusion of sets play a major role. The idea of De Baets ([2]) was to fuzzify the underlying logical operations, i.e. the Boolean conjuntion and the Boolean implication, to obtain a successful fuzzification. An n-dimensional gray-scale image is model as an $I\!R^n \to [0,1]$ function. It is required that the gray values of the image belong to the real unit interval in order to consider an image as a fuzzy object. Taking two n-dimensional images A and B, a conjunctor $\mathcal{C}$ and an implicator $\mathcal{I}$, we have the following definitions:

Definition 6. *The fuzzy dilation $D_{\mathcal{C}}(A,B)$ and fuzzy erosion $E_{\mathcal{I}}(A,B)$ of A by B are the gray-scale images defined by*

$$D_{\mathcal{C}}(A,B)(y) = \sup_x \mathcal{C}(B(x-y), A(x)) \quad (1)$$

$$E_{\mathcal{I}}(A,B)(y) = \inf_x \mathcal{I}(B(x-y), A(x)). \quad (2)$$

Definition 7. *The fuzzy closing $C_{\mathcal{C},\mathcal{I}}(A,B)$ and fuzzy opening $O_{\mathcal{C},\mathcal{I}}(A,B)$ of A by B are the gray-scale images defined by*

$$C_{\mathcal{C},\mathcal{I}}(A,B)(y) = E_{\mathcal{I}}(D_{\mathcal{C}}(A,B),-B)(y) \quad (3)$$

$$O_{\mathcal{C},\mathcal{I}}(A,B)(y) = D_{\mathcal{C}}(E_{\mathcal{I}}(A,B),-B)(y). \quad (4)$$

Note that the reflection $-B$ of a n-dimensional fuzzy set B is defined by $-B(x) = B(-x)$, for all $x \in I\!R^n$.

Obviously, we can use conjunctive uninorm and related implicators to define fuzzy morphological operators following the previous definitions. We investigate in [14] which conjunctive uninorms need to be chosen in order to preserve the algebraic and morphological properties satisfied by the classical morphological operators. Moreover, going so far than De Bates in [9], it is given in ([14]) sufficient and/or necessary conditions on the conjunctive uninorms in order to guarantee these properties.

Given a strong negation $\mathcal{N}$, we define by $(co_{\mathcal{N}}A)(x) = \mathcal{N}(A(x))$ the $\mathcal{N}$-complement $co_{\mathcal{N}}A$ of a fuzzy set A. Two fuzzy morphological operations P and Q are called $\mathcal{N}$-dual if for any two gray-scale objects A and B it holds that $P(A,B) = co_{\mathcal{N}}Q(co_{\mathcal{N}}A, B)$.

All results in this paper are concerning to a left-continuous conjunctive uninorm and its residual implicator $\mathcal{I}_{\mathcal{U}}$. However, it is known that the fuzzy dilation and fuzzy erosion are $\mathcal{N}$-dual if and only if $\mathcal{I} = \mathcal{I}_{\mathcal{C},\mathcal{N}}$ (or equivalently $\mathcal{C} = \mathcal{C}_{\mathcal{I},\mathcal{N}}$), moreover, if the fuzzy dilation and fuzzy erosion are $\mathcal{N}$-dual, then also the fuzzy closing and fuzzy opening are $\mathcal{N}$-dual [2]. Hence, to have duality between our fuzzy morphological operators, we need to use conjunctive uninorms satisfying

$$\mathcal{I}_{\mathcal{U}} = \mathcal{I}_{\mathcal{U},\mathcal{N}}.$$

This property always occurs for two special kinds of uninorms given in the following proposition (see [6] and [13]).

Proposition 3.1. *The identity $\mathcal{I}_{\mathcal{U}} = \mathcal{I}_{\mathcal{U},\mathcal{N}}$ is satisfied in each one of the following situations*

i) When $\mathcal{U}$ is a conjunctive representable uninorm and $\mathcal{N}$ is the strong negation obtained from the additive generator h of $\mathcal{U}$ by $\mathcal{N}(a) = h^{-1}(-h(a))$.

ii) When $\mathcal{N}$ is any strong negation and $\mathcal{U}$ is the corresponding conjunctive, left-continuous, idempotent uninorm $\mathcal{U}^{\mathcal{N}}$.

Thus, these two kinds of conjunctive uninorms guarantee duality between fuzzy morphological operators. Consequently, they are the most suitable in our framework.

We summarized in the following the algebraic properties of the fuzzy morphological operators needed in the next section (see [14], [11], for a detailed account of these).

Proposition 3.2. *Let $\mathcal{U}$ be a left-continuous, conjunctive uninorm and $I_{\mathcal{U}}$ its residual implicator. Let A_1 and A_2 be two gray-scale images and let B be a gray-scale structuring element. Then it holds:*

a) $E_{I_{\mathcal{U}}}$, $D_{\mathcal{U}}$, $C_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}$ and $O_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}$ are increasing in the first place.

b) Moreover, they satisfy:

$$E_{I_{\mathcal{U}}}(A_1 \cap A_2, B) = E_{I_{\mathcal{U}}}(A_1, B) \cap E_{I_{\mathcal{U}}}(A_2, B)$$

$$E_{I_{\mathcal{U}}}(A_1 \cup A_2, B) \supseteq E_{I_{\mathcal{U}}}(A_1, B) \cup E_{I_{\mathcal{U}}}(A_2, B)$$

$$D_{\mathcal{U}}(A_1 \cup A_2, B) = D_{\mathcal{U}}(A_1, B) \cup D_{\mathcal{U}}(A_2, B)$$

$$D_{\mathcal{U}}(A_1 \cap A_2, B) \subseteq D_{\mathcal{U}}(A_1, B) \cap D_{\mathcal{U}}(A_2, B).$$

The extensivity of the fuzzy dilation and the anti-extensivity of the fuzzy morphological operators are ensured by next propositions.

Proposition 3.3. *Let $\mathcal{U}$ be a conjunctive uninorm with neutral element $e \in (0,1)$, let $I_{\mathcal{U}}$ be its residual implicator and let B be a gray-scale structuring element such that $B(0) = e$. Then the following inclusions hold:*

$$E_{\mathcal{I}_{\mathcal{U}}}(A, B) \subseteq A \subseteq D_{\mathcal{U}}(A, B).$$

Proposition 3.4. *Let $\mathcal{U}$ be a left-continuous conjunctive uninorm and $I_{\mathcal{U}}$ its residual implicator, let A be a gray-scale image and let B be a gray-scale structuring element, then it holds*

1. *The fuzzy closing $C_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}$ is extensive and the fuzzy opening is anti-extensive:*

$$O_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(A, B) \subseteq A \subseteq C_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(A, B).$$

2. *The fuzzy closing and the fuzzy opening are idempotent, i.e.:*

$$C_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(C_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(A, B), B) = C_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(A, B),$$
$$O_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(O_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(A, B), B) = O_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(A, B).$$

Proposition 3.5. *Let $\mathcal{U}$ be a conjunctive uninorm, with neutral element $e \in (0,1)$ and let $I_{\mathcal{U}}$ be its residual implicator. Let A be a gray* *scale-image and let B be a gray-scale structuring element such that $B(0) = e$, then it holds that*

$$\begin{aligned} E_{\mathcal{I}_\mathcal{U}}(A,B) \subseteq & O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B) \subseteq A \\ & \subseteq C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B) \subseteq D_\mathcal{U}(A,B). \end{aligned}$$

4 Closed and open fuzzy objects

The idempotence of the fuzzy closing and opening when we get $\mathcal{U}$ a left-continuous conjunctive uninorm and $I_\mathcal{U}$ its residual implicator motivate, as in the classical mathematical morphology, the following definitions.

Definition 8. *Let A and B two gray-scale images We say that A is B-closed (resp. B-open) if $C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B) = A$ (resp. $O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B) = A$).*

Observe that, as a consequence of Prop. 3.4, $C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B)$ is B-closed and $O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B)$ is B-open. Moreover, we have the following proposition that was advanced by De Baets in [15] without proof. We include it for the sake of clarity.

Proposition 4.1. *Let $\mathcal{U}$ be a left-continuous, conjunctive uninorm and $I_\mathcal{U}$ its residual implicator. Then it holds:*

a) A is B-open if and only if there exists a fuzzy object F such that $A = D_\mathcal{U}(F, -B)$.

b) A is B-closed if and only if there exists a fuzzy object F such that $A = E_{I_\mathcal{U}}(F, -B)$.

Proof. Let us assume that A is B-open. By definition of fuzzy opening, choosing $F = E_{\mathcal{I}_\mathcal{U}}(A,B)$ we have a fuzzy object satisfying that $D_\mathcal{U}(F, -B) = A$. Now suppose that A can be represented as $A = D_\mathcal{U}(F, -B)$ for some fuzzy object F. From Prop. 3.4 we know that $O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B) \subseteq A$. In order to prove the other inclusion, using Prop. 2.5 and that $\mathcal{I}_\mathcal{U}$ is increasing in the second partial map, we have:

$$\begin{aligned} & E_{I_\mathcal{U}}(A,B)(y) = \inf_x I_\mathcal{U}(B(x-y), A(x)) \\ & = \inf_x I_\mathcal{U}(B(x-y), D_\mathcal{U}(F,-B)(x)) \\ & = \inf_x I_\mathcal{U}(B(x-y), \sup_z \mathcal{U}(B(x-z), F(z))) \\ & \geq \inf_x I_\mathcal{U}(B(x-y), \mathcal{U}(B(x-y), F(y))) \\ & \geq F(y). \end{aligned}$$

So, we have shown that $F \subseteq E_{I_\mathcal{U}}(A,B)$. Then, by Prop. 3.2 we have that $A \subseteq O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B)$. A similar argument proves b). □

As it was pointed out by Bodenhofer in [16] opening and closing operators only make sense if the opening always gives an open result, and the closing operator gives a closed result. Moreover, it is desirable to have "extremal properties". We see now that this last requirement is also satisfied by our opening and closing fuzzy operators.

Proposition 4.2. *Let $\mathcal{U}$ be a left-continuous, conjunctive uninorm and $I_\mathcal{U}$ its residual implicator. Then the following holds:*

a) $O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B)$ is the largest B-open fuzzy subset of A.

b) $C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B)$ is the smallest B-closed fuzzy superset of A.

Proof. **a)** We now that $O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B)$ is B-open, and, from proposition 3.4 that $O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B) \subseteq A$. Now let us assume that $E \subseteq A$ and E is B-open. Then, as E is B-open we have $E = O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(E,B)$. By proposition 3.2 $O_{\mathcal{U},\mathcal{I}_\mathcal{U}}$ is increasing in the first argument, then $E = O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(E,B) \subseteq O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B)$.

b) We now that $C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B)$ is B-closed and $A \subseteq C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B)$. Let E a B-closed superset of A, $A \subseteq E$. Also, by proposition 3.2 we know that $C_{\mathcal{U},\mathcal{I}_\mathcal{U}}$ is increasing in the first argument, then

$$C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B) \subseteq C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(E,B) = E,$$

because, E is B-closed. Therefore, $C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A,B)$ must be the smallest B-closed fuzzy superset of A.

□

Now let us briefly consider the preservation of B-openness and B-closedness by intersections and unions, respectively.

Proposition 4.3. *Let $\mathcal{U}$ be a left-continuous, conjunctive uninorm and $I_\mathcal{U}$ its residual implicator. Let A_1 and A_2 be two gray-levels images and B a gray-scale structuring element. Then, it holds:*

a) If A_1 and A_2 are both B-open then, $A_1 \cup A_2$ is B-open.

b) If A_1 and A_2 are both B-closed then, $A_1 \cap A_2$ is B-closed.

Proof. **a)** If A_1 and A_2 are B-open then, $O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A_1, B) = A_1$ and $O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A_2, B) = A_2$. By proposition 3.2 we have that

$$\begin{aligned}
&A_1 \cup A_2 = O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A_1, B) \cup O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A_2, B)\\
&= D_\mathcal{U}(E_{\mathcal{I}_\mathcal{U}}(A_1, B), -B) \cup D_\mathcal{U}(E_{\mathcal{I}_\mathcal{U}}(A_2, B), -B)\\
&= D_\mathcal{U}(E_{\mathcal{I}_\mathcal{U}}(A_1, B) \cup E_{\mathcal{I}_\mathcal{U}}(A_2, B), -B)\\
&\subseteq D_\mathcal{U}(E_{\mathcal{I}_\mathcal{U}}(A_1 \cup A_2, B), -B)\\
&= O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A_1 \cup A_2, B),
\end{aligned}$$

thus $A_1 \cup A_2 \subseteq O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A_1 \cup A_2, B)$. The other inclusion is a consequence of proposition 3.4. Therefore, $A_1 \cup A_2 = O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A_1 \cup A_2, B)$ and $A_1 \cup A_2$ is a B-open fuzzy set.

b) Let us assume now that A_1 and A_2 are B-closed fuzzy sets. Thus, $A_1 = C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A_1, B)$ and $A_2 = C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A_2, B)$. Then, using proposition 3.2 we obtain the following:

$$\begin{aligned}
&A_1 \cap A_2 = C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A_1, B) \cap C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A_2, B)\\
&= E_{\mathcal{I}_\mathcal{U}}(D_\mathcal{U}(A_1, B), -B) \cap E_{\mathcal{I}_\mathcal{U}}(D_\mathcal{U}(A_2, B), -B)\\
&= E_{\mathcal{I}_\mathcal{U}}(D_\mathcal{U}(A_1, B) \cap D_\mathcal{U}(A_2, B), -B)\\
&\supseteq E_{\mathcal{I}_\mathcal{U}}(D_\mathcal{U}(A_1 \cap A_2, B), -B)\\
&= C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A_1 \cap A_2, B).
\end{aligned}$$

Thus, $A_1 \cap A_2 \supseteq C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A_1 \cap A_2, B)$ and therefore, using proposition 3.4, $A_1 \cap A_2 = C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A_1 \cap A_2, B)$ and thus $A_1 \cap A_2$ is a B-closed fuzzy set. □

The previous propositions are valid for any left-continuous conjunctive uninorm. However, if we want to have duality between closed and open fuzzy objects we again need the two kins of uninorms stated in Prop. 3.1.

Proposition 4.4. *Let $\mathcal{U}$ be a conjunctive uninorm satisfying the condition i) or ii) from proposition 3.1 then, A is B-open if and only if the $\mathcal{N}$-complement $co_\mathcal{N} A$ of A is B-closed.*

Proof. We will prove the implication from right to left. Assume that $co_\mathcal{N} A$ is B-closed, $C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(co_\mathcal{N} A, B) = co_\mathcal{N} A$. Complementing both sides, since $\mathcal{N}$ is an involutive operator, and closing and opening are $\mathcal{N}$-dual, then we obtain

$$O_{\mathcal{U},\mathcal{I}_\mathcal{U}}(A, B) = co_\mathcal{N} C_{\mathcal{U},\mathcal{I}_\mathcal{U}}(co_\mathcal{N} A, B) = A$$

thus A is B-open. The converse holds similarly □

Let us now prove that, using representable uninorms like in Prop. 3.1, we obtain the so-called generalized idempotence laws for fuzzy closing and fuzzy opening. First we need several results.

Proposition 4.5. *Let $\mathcal{U}$ be a conjunctive representable uninorm with additive generator h. For all $a, b, c, d, x, y \in [0, 1]$, if*

$$\mathcal{U}(a, \mathcal{I}_\mathcal{U}(b, c)) \geq d, \quad \mathcal{U}(x, \mathcal{I}_\mathcal{U}(y, b)) \geq a$$

then

$$\mathcal{U}(x, \mathcal{I}_\mathcal{U}(y, c)) \geq d.$$

Proof. Cases when $\mathcal{U}$ or $\mathcal{I}_\mathcal{U}$ admit no representation through the generator h can be easily verified separately. In any other case we have:

$$\mathcal{U}(a, \mathcal{I}_\mathcal{U}(b, c)) = h^{-1}(h(a) - h(b) + h(c)) \geq d$$

and

$$\mathcal{U}(x, \mathcal{I}_\mathcal{U}(y, b)) = h^{-1}(h(x) - h(y) + h(b)) \geq a.$$

Since h is increasing, by applying h to the previous inequalities and adding them we obtain

$$h(c) + h(x) - h(y) \geq h(d)$$

and the proposition follows. □

Proposition 4.6. *Let $\mathcal{U}$ be a conjunctive representable uninorm with additive generator h. For all $a, b, c, d, x, y, z, t \in [0, 1]$, if*

$$\mathcal{U}(a, \mathcal{I}_\mathcal{U}(b, c)) \geq d, \quad \mathcal{U}(c, \mathcal{I}_\mathcal{U}(x, y)) \leq z$$

$$\textit{and} \quad \mathcal{U}(d, \mathcal{I}_\mathcal{U}(x, y)) \geq t, \quad \textit{then}$$

$$\mathcal{U}(a, \mathcal{I}_\mathcal{U}(b, z)) \geq t.$$

Proof. It is similar to the previous one. □

Figure 1: Input image used in the experiments

The proof of inclusions concerning fuzzy opening in the following two propositions are quite similar to those given by De Bates in [15] for continuous t-norms, using in our case Prop. 4.5 and Prop. 4.6. With respect to inclusions concerning fuzzy closing, their proofs follow from duality, guaranteed by Prop. 3.1.

Proposition 4.7. *Let $\mathcal{U}$ be a conjunctive representable uninorm. If A is B-open and rang(A) and rang(B) are finite sets, then for any fuzzy object F it holds:*

$$O_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(F,A) \subseteq O_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(F,B) \subseteq F$$

and dually

$$F \subseteq C_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(F,B) \subseteq C_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(F,A)$$

Proposition 4.8. *Let $\mathcal{U}$ be a conjunctive representable uninorm. If A is B-open and rang(A) and rang(B) are finite sets, then for any fuzzy object F it holds:*

$$\begin{aligned} &O_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(O_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(F,B),A) \\ &\quad = O_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(O_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(F,A),B) = O_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(F,A) \end{aligned}$$

and dually

$$\begin{aligned} &C_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(C_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(F,B),A) \\ &\quad = C_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(C_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(F,A),B) = C_{\mathcal{U},\mathcal{I}_{\mathcal{U}}}(F,A) \end{aligned}$$

5 Experimental results

In this section we present some experiments showing the differences between basic fuzzy morphological operators using different uninorms. The examples presented illustrate the influence of the choice of the pair $(\mathcal{U},\mathcal{I}_{\mathcal{U}})$ using both, idempotent and representable conjunctive uninorms. Our input image, A, is depicted in Fig. 1. The structuring element, B, used for the "fuzzy" operators is represented by the matrix

$$B = e \cdot \begin{pmatrix} 0.86 & 0.86 & 0.86 \\ 0.86 & 1.00 & 0.86 \\ 0.86 & 0.86 & 0.86 \end{pmatrix} \tag{5}$$

where e is the neutral element of the uninorm.

In Fig. 2 we show, from left to right, the fuzzy dilation, fuzzy erosion and the fuzzy gradient operator, $D_{\mathcal{U}}(A,B) \setminus E_{\mathcal{I}_{\mathcal{U}}}(A,B)$, using several idempotent uninorms. ¿From top to bottom we have used $\mathcal{U}^{\mathcal{N}}$ and $\mathcal{I}_{\mathcal{U}^{\mathcal{N}}}$ where $\mathcal{N}(x) = \sqrt{1-x^2}$ and $\mathcal{N}(x) = 1-x$ respectively. The corresponding neutral elements are, $e = 1/\sqrt{2}$ and $e = 0.5$.

Fig. 3 have the same structure that the previous one. There, we use two different representable conjunctive uninorms with the same neutral element $e = 0.5$, and the Lukasiewicz t-norm T_L, respectively. From top to bottom, we use, $h(x) = \ln\left(\frac{x}{1-x}\right)$, $h_e(x) = \frac{x-e}{x(1-x)}$, and the pair (T_L, I_{T_L}) with the structuring element obtained taking $e = 1$ in (5). While the hard edges are detected very well in all the cases, it can be observed that the edge-images obtained using conjunctive uninorms detect soft edges better than the edge-image with (T_L, I_{T_L}). Remark that the pair (T_L, I_{T_L}) is a pair of t-norm and implicator that guarantees the fulfillment of all the properties in order to have a good fuzzy mathematical morphology ([11]).

Acknowledgements

The first author has been partially subsidized by CICYT under grant TIC2001-0931. The other two authors have been partially supported by the Government of the Balearic Islands grant number PDIB-2002GC3-19.

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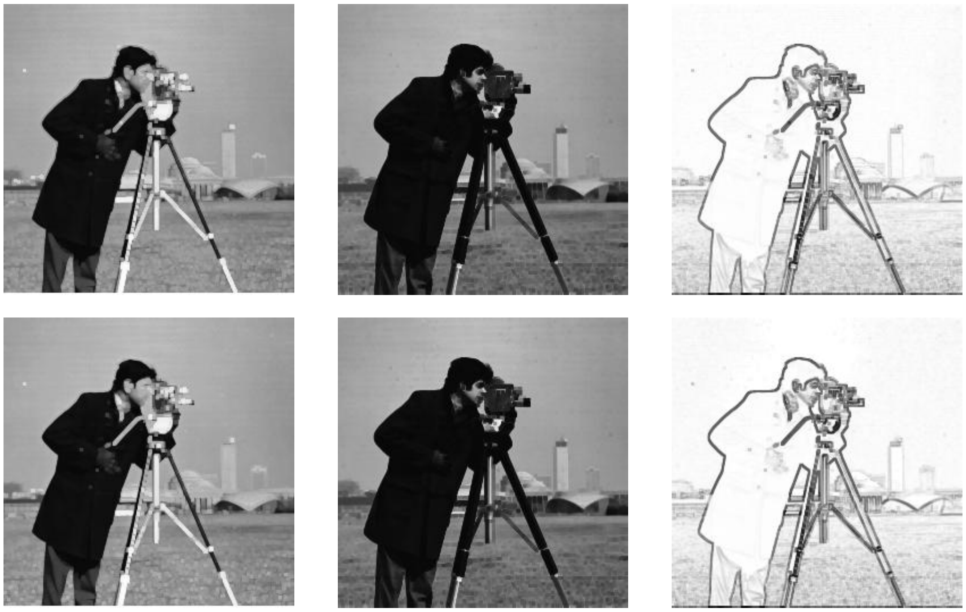

Figure 2: From top to bottom, dilation, erosion and gradient obtained using idempotent uninorms with negation $\mathcal{N}(x) = \sqrt{1-x^2}$ and $\mathcal{N}(x) = 1-x$, respectively

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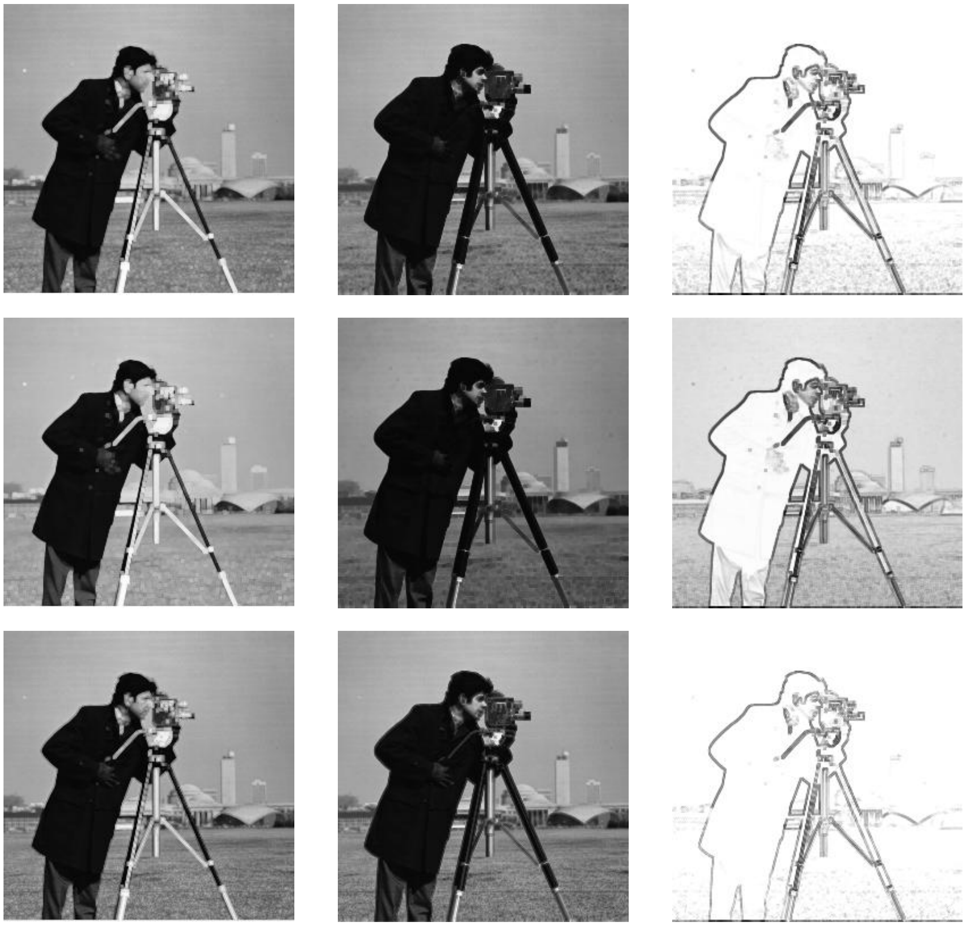

Figure 3: Fuzzy dilation, fuzzy erosion and fuzzy gradient obtained using representable conjunctive uninorms with additive generator $h(x) = \ln(x/(1-x))$, $h(x) = (x-0.5)/(x(1-x))$, and the pair (T_L, I_{T_L})

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