

Algebraic Properties of Fuzzy Morphological Operators based on Uninorms

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Abstract. In this paper, an approach to fuzzy mathematical morphology based on conjunctive uninorms is studied. It is proved that the most suitable conjunctive uninorms to be used in this framework are two special kinds of both, representable and idempotent uninorms. For these operators, it is proved that the most usual algebraic and morphological properties are preserved, such as, duality, monotonicity, interaction with union and intersection, invariance under translating and scaling, local knowledge property, extensivity, idempotence, and many others.

Keywords: Fuzzy mathematical morphology, erosion, dilation, uninorms, representable and idempotent uninorms, implicators.

1 Introduction

The identification of objects, object feature extraction and anomalies detection in automated industrial processes are closely connected with the recognition of shapes and therefore with the recognition or vision systems. In this context the mathematical morphology is a useful tool for extraction image components that are useful in the representation and description of region shapes, such as boundaries, skeletons, and convex hull. The basic tools of mathematical morphology are the morphological operations. A morphological operation P transforms the structure that we want to analyze A (an image) by means of a small object B , called “structuring element”, with we want to probe the structure of A , into a new object $P(A, B)$ (a new image). The basic morphological operations are the dilation and erosion. These operations are based on set theory and were originally developed for binary images (black and white) and afterwards successfully extended to gray-scale images.

Nevertheless, the shapes in an image are not always crisply defined, and uncertainty can arise within each level of image analysis and pattern recognition. It can occur at the low-level in the raw sensor output, and it can be extended all the way through intermediate and higher levels. A recognition or computer vision system must have sufficient flexibility for

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processing the uncertainty in any of these levels, so that the system could retain as much of the information content of the data as possible, at each level. As the first essential step of a recognition or vision system is the feature extraction, the method used should have a provision for representing and manipulating the uncertainties. Fuzzy set theory provides a mechanism to represent and manipulate uncertainty and ambiguity. Fuzzy operators and their properties as well as fuzzy inference rules have found considerable applications in image analysis and pattern recognition (see [11],[2] and references therein).

In order to do that, the fuzzy mathematical morphology is an alternative extension of binary morphology to gray-scale morphology using concepts and techniques from fuzzy set theory. Several researchers have introduced alternative morphological operations. Bloch and Maître [1] follow an approach using t-norms and the associated model implicator, with an involutive negator. Fuzzy set inclusion was used by Zadeh, Sinha & Dougherty, Kitainik and Bandler & Kohout in order to define fuzzy morphological operators. The “Minkowsky addition” was initially used by De Baets et al. in [7],[8]. A detailed account can be found in [11], [2] and references therein. In this paper we focus our attention in the general framework for fuzzy mathematical morphology constructed by De Baets in [3] where he uses “conjunctors” and “implicators” in order to define “fuzzy erosion” and “fuzzy dilation”, without forcing duality relationships between these operators, and obtaining good properties for the corresponding fuzzy closing and fuzzy opening operators.

Our goal is to study this general framework when we use conjunctive uninorms. A first tentative appears in [9] where De Baets et al. use conjunctive uninorms and its residual implicators to construct the fuzzy morphological operators. Taking into account that duality and idempotency are indispensable for the further development of fuzzy mathematical morphology, it is proved in [9] that left-continuous, conjunctive, representable uninorms are suitable to be used in this framework. But there is another class of conjunctive uninorm being also suitable in this way as we prove here: some idempotent ones. Then, the aim of this paper is the construction of a fuzzy morphology based on conjunctive uninorms, including the representable and the idempotent ones, taking into account the morphological and algebraic properties of the fuzzy morphological operators and going so far than De Baets in [9].

The paper is organized as follows. In the next Section we review the basic definitions and properties of fuzzy logical operators needed in the subsequent sections. In Section 3, we present the general framework initiated by De Baets in [3]. We will discuss, in Section 4, the algebraic and morphological properties satisfied by the fuzzy morphological operators based on left-continuous conjunctive uninorms, following a similar structure than the used in Chapter 1 of [11]. The paper ends with some conclusions and the future work.

2 Fuzzy logical operators

Let us recall the fuzzy logical operators that we will use throughout the paper. More details on these operators can be found for instance in [12].

Definition 1. *A decreasing and involutive unary operator \mathcal{N} on $[0, 1]$ with $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$ is called a strong negation.*

Definition 2. *An increasing binary operator \mathcal{C} on $[0, 1]$ is called a conjunctor if it satisfies*

$$\mathcal{C}(0, 1) = \mathcal{C}(1, 0) = 0 \quad \text{and} \quad \mathcal{C}(1, 1) = 1.$$

Definition 3. A binary operator \mathcal{I} on $[0, 1]$ is called an *implicator* if it is decreasing with the first partial map, increasing with the second one and it satisfies

$$\mathcal{I}(0, 0) = \mathcal{I}(1, 1) = 1 \quad \text{and} \quad \mathcal{I}(1, 0) = 0.$$

One can construct conjunctors and implicators from each other. On one hand, given an implicator \mathcal{I} and a strong negation \mathcal{N} the binary operator defined by

$$\mathcal{C}_{\mathcal{I}, \mathcal{N}}(a, b) = \mathcal{N}(\mathcal{I}(a, \mathcal{N}(b)))$$

is a conjunctor. On the other hand, given a conjunctor \mathcal{C} and a strong negation \mathcal{N} the binary operator defined by

$$\mathcal{I}_{\mathcal{C}, \mathcal{N}}(a, b) = \mathcal{N}(\mathcal{C}(a, \mathcal{N}(b)))$$

is an implicator. Another way to construct implicators from conjunctors is by residuation. Given a conjunctor \mathcal{C} the binary operator

$$\mathcal{I}_{\mathcal{C}}(a, b) = \sup\{c \in [0, 1] \mid \mathcal{C}(a, c) \leq b\}$$

is an implicator called the residual implicator of \mathcal{C} .

A special kind of conjunctors is given by the well known t-norms. In fact, fuzzy morphological operators are usually constructed from t-norms and, a special kind of them, the nilpotent ones, has been proved to be the most useful in this framework (see for instance [11]). However, a generalization of t-norms has appeared and has been studied in [10]:

Definition 4. (See [10]) A *uninorm* is a two-place function $U : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ which is associative, commutative, increasing in each place and such that there exists some element $e \in [0, 1]$, called the *neutral element*, such that $U(e, x) = x$ for all $x \in [0, 1]$.

It is clear that the function U becomes a t-norm when $e = 1$ and a t-conorm when $e = 0$. For any uninorm we have $U(0, 1) \in \{0, 1\}$ and a uninorm U is said conjunctive when $U(1, 0) = 0$ and disjunctive when $U(1, 0) = 1$. Moreover, a uninorm U is said to be idempotent whenever $U(x, x) = x$ for all $x \in [0, 1]$.

This kind of operators results specially interesting because of their behavior: like a t-norm in $[0, e]^2$ and like a t-conorm in $[e, 1]^2$.

There are three known classes of conjunctive uninorms ([4]): uninorms in \mathcal{U}_{\min} , representable uninorms and idempotent uninorms. The first two classes have been already used in fuzzy morphology in [9]. Since left-continuity is essential in order to have “good” properties, and there is no left-continuous uninorms in the class \mathcal{U}_{\min} , we will only use here representable and idempotent, conjunctive uninorms. Of course, a fuzzy morphology can be done using also uninorms in \mathcal{U}_{\min} , but all properties stated and proved in this paper where left-continuity is required can fail, for this kind of logical operators. Let us recall here the definitions and characterizations of representable and idempotent uninorms, but more details of these classes can be found in [10] and [5] respectively.

Definition 5. Let $e \in (0, 1)$ and let $h : [0, 1] \rightarrow [-\infty, +\infty]$ be a strictly increasing, continuous function with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$. The binary operator \mathcal{U} defined by

$$\mathcal{U}(a, b) = h^{-1}(h(a) + h(b))$$

for all $(a, b) \in [0, 1]^2 \setminus \{(0, 1), (1, 0)\}$ and $\mathcal{U}(0, 1) = \mathcal{U}(1, 0) = 0$ is a conjunctive uninorm with neutral element e . This kind of uninorms are usually called *representable uninorms*.

Theorem 2.1. *A conjunctive uninorm \mathcal{U} with neutral element $e \in (0, 1)$ is representable if and only if it is strictly increasing and continuous on $(0, 1)^2$ and there is a strong negation \mathcal{N} with $\mathcal{N}(e) = e$, such that for any $(a, b) \in [0, 1]^2 \setminus \{(0, 1), (1, 0)\}$*

$$\mathcal{U}(a, b) = \mathcal{N}(\mathcal{U}(\mathcal{N}(a), \mathcal{N}(b))).$$

Theorem 2.2. *Let \mathcal{U} be a representable uninorm with additive generator h , then its residual implicator $I_{\mathcal{U}}$ is given by*

$$I_{\mathcal{U}}(x, y) = \begin{cases} h^{-1}(h(y) - h(x)) & \text{if } (x, y) \in [0, 1]^2 \setminus \{(0, 0), (1, 1)\} \\ 1 & \text{otherwise} \end{cases}$$

Following with the idea to have the “good” properties derived from left-continuity, we will only use in this paper left-continuous, conjunctive, idempotent uninorms. However, note again that any other kind of conjunctive, idempotent uninorms (see [13]) can also be used in the same way.

Theorem 2.3. *A binary operator \mathcal{U} is a left-continuous idempotent uninorm with neutral element $e \in (0, 1)$ if and only if there exists a decreasing function $g : [0, 1] \rightarrow [0, 1]$ with fix point e , satisfying $g^2(x) \geq x$ for all $x \leq g(0)$ and $g(x) = 0$ for all $x > g(0)$ such that, for all $x, y \in [0, 1]$, \mathcal{U} is given by*

$$\mathcal{U}(x, y) = \begin{cases} \min(x, y) & \text{if } y \leq g(x) \text{ and } x \leq g(0) \\ \max(x, y) & \text{elsewhere} \end{cases}$$

Note that given any strong negation \mathcal{N} we obtain a left-continuous idempotent uninorm just taking $g = \mathcal{N}$, that we will denote by $\mathcal{U}^{\mathcal{N}}$. On the other hand, the residual implicator of an idempotent uninorm is given by

Theorem 2.4. *Let \mathcal{U} be any idempotent uninorm with $g(0) = 1$. The residual implicator $I_{\mathcal{U}}$ is given by:*

$$I_{\mathcal{U}}(x, y) = \begin{cases} \min(g(x), y) & \text{if } y < x \\ \max(g(x), y) & \text{if } y \geq x \end{cases}$$

Proposition 2.5. *(See [6]) Let \mathcal{U} be a conjunctive uninorm and $I_{\mathcal{U}}$ its residual implicator.*

- *The second partial map of $I_{\mathcal{U}}$ is right-continuous and for all $x, y \in [0, 1]$,*

$$y \leq I_{\mathcal{U}}(x, \mathcal{U}(x, y)).$$

- *If \mathcal{U} is left-continuous then so is the first partial map of $I_{\mathcal{U}}$, $I_{\mathcal{U}}$ satisfies the exchange principle:*

$$I_{\mathcal{U}}(x, I_{\mathcal{U}}(y, z)) = I_{\mathcal{U}}(y, I_{\mathcal{U}}(x, z)),$$

and also the following properties:

$$\mathcal{U}(x, I_{\mathcal{U}}(x, y)) \leq y, \quad I_{\mathcal{U}}(\mathcal{U}(x, y), z) = I_{\mathcal{U}}(x, I_{\mathcal{U}}(y, z))$$

for all $x, y, z \in [0, 1]$.

3 Fuzzy morphological operators

From the definition of classical erosion and dilation ([11]) it is clear that the intersection and inclusion of sets play a major role. The idea of De Baets ([3]) was to fuzzify the underlying logical operations, i.e. the Boolean conjunction and the Boolean implication, to obtain a successful fuzzification. An n -dimensional gray-scale image is model as an $\mathbb{R}^n \rightarrow [0, 1]$ function. It is required that the gray values of the image belong to the real unit interval in order to consider an image as a fuzzy object. Taking two n -dimensional images A and B , a conjunctor \mathcal{C} and an implicator \mathcal{I} , we have the following definitions:

Definition 6. *The fuzzy dilation $D_{\mathcal{C}}(A, B)$ and fuzzy erosion $E_{\mathcal{I}}(A, B)$ of A by B are the gray-scale images defined by*

$$D_{\mathcal{C}}(A, B)(y) = \sup_x \mathcal{C}(B(x - y), A(x)) \quad (1)$$

$$E_{\mathcal{I}}(A, B)(y) = \inf_x \mathcal{I}(B(x - y), A(x)). \quad (2)$$

Definition 7. *The fuzzy closing $C_{\mathcal{C}, \mathcal{I}}(A, B)$ and fuzzy opening $O_{\mathcal{C}, \mathcal{I}}(A, B)$ of A by B are the gray-scale images defined by*

$$C_{\mathcal{C}, \mathcal{I}}(A, B)(y) = E_{\mathcal{I}}(D_{\mathcal{C}}(A, B), -B)(y) \quad (3)$$

$$O_{\mathcal{C}, \mathcal{I}}(A, B)(y) = D_{\mathcal{C}}(E_{\mathcal{I}}(A, B), -B)(y). \quad (4)$$

Note that the reflection $-B$ of a n -dimensional fuzzy set B is defined by $-B(x) = B(-x)$, for all $x \in \mathbb{R}^n$.

Obviously, since a conjunctive uninorm is a conjunctor, we can use conjunctive uninorm and related implicators to define fuzzy morphological operators following the previous definitions. Next, we investigate which conjunctive uninorms need to be chosen in order to preserve the algebraic and morphological properties, such as, duality, monotonicity, interaction with union and intersection, invariance under translating and scaling, extensivity and idempotence, inclusion properties, commutativity and associativity of the fuzzy dilation, combinations of dilation and erosion, local knowledge property and adjunction property.

4 Algebraic properties of morphological operators using idempotent uninorms

In this section we will give sufficient and/or necessary conditions on the conjunctive uninorms in order to guarantee similar properties as in the binary and gray-scale mathematical morphology.

Given a strong negation \mathcal{N} , we define by $(co_{\mathcal{N}}A)(x) = \mathcal{N}(A(x))$ the \mathcal{N} -complement $co_{\mathcal{N}}A$ of a fuzzy set A . Two fuzzy morphological operations P and Q are called \mathcal{N} -dual if for any two gray-scale objects A and B it holds that $P(A, B) = co_{\mathcal{N}}Q(co_{\mathcal{N}}A, B)$.

All results in this paper are concerning to a left-continuous conjunctive uninorm and its residual implicator $\mathcal{I}_{\mathcal{U}}$. However, it is known that the fuzzy dilation and fuzzy erosion are \mathcal{N} -dual if and only if $\mathcal{I} = \mathcal{I}_{\mathcal{C}, \mathcal{N}}$ (or equivalently $\mathcal{C} = \mathcal{C}_{\mathcal{I}, \mathcal{N}}$), moreover, if the fuzzy dilation and fuzzy erosion are \mathcal{N} -dual, then also the fuzzy closing and fuzzy opening are \mathcal{N} -dual [3]. Hence, to have duality between our fuzzy morphological operators, we need to use conjunctive uninorms satisfying

$$\mathcal{I}_{\mathcal{U}} = \mathcal{I}_{\mathcal{U}, \mathcal{N}}.$$

This property always occurs for two special kinds of uninorms given in the following proposition (see [6] or [13]).

Proposition 4.1. *The identity*

$$\mathcal{I}_{\mathcal{U}} = \mathcal{I}_{\mathcal{U}, \mathcal{N}}$$

is satisfied in each one of the following situations

i) When \mathcal{U} is a conjunctive representable uninorm and \mathcal{N} is the strong negation obtained from the additive generator h of \mathcal{U} by

$$\mathcal{N}(a) = h^{-1}(-h(a)).$$

ii) When \mathcal{N} is any strong negation and \mathcal{U} is the corresponding conjunctive, left-continuous, idempotent uninorm $\mathcal{U}^{\mathcal{N}}$.

Thus, these two kinds of conjunctive uninorms are the most suitable in our framework.

Note that next propositions (from 4.2 to 4.8) are actually particular cases of those stated, for instance in [11]. Thus either, we do not include their proof or we only note that the used class of uninorms satisfies the properties required on the conjunctive.

Proposition 4.2. *Let \mathcal{U} be a conjunctive uninorm and $I_{\mathcal{U}}$ its residual implicator. It holds: the fuzzy dilation $D_{\mathcal{U}}$ is increasing in both arguments, the fuzzy erosion $E_{\mathcal{I}}$ is increasing in the first argument and decreasing in the second one, the fuzzy closing $C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}$ and the fuzzy opening $O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}$ are both increasing in the first argument.*

The following two propositions concern interaction properties with Zadeh's union and intersection. For an arbitrary family $(A_i)_{i \in I}$ of fuzzy sets the Zadeh union and intersection are defined by: $\cup_{i \in I} A_i(x) = \sup_{i \in I} A_i(x)$ and $\cap_{i \in I} A_i(x) = \inf_{i \in I} A_i(x)$.

Proposition 4.3. *Let \mathcal{U} be a left-continuous, conjunctive uninorm and $I_{\mathcal{U}}$ its residual implicator. Let A be a gray scale image and let B be a gray-scale structuring element. Further, let $(A_i)_{i \in I}$ be an arbitrary family of gray-scale images and let $(B_i)_{i \in I}$ be an arbitrary family of gray-scale structuring elements. Then it holds*

$$\begin{aligned} D_{\mathcal{U}}\left(\bigcup_{i \in I} A_i, B\right) &= \bigcup_{i \in I} D_{\mathcal{U}}(A_i, B), & D_{\mathcal{U}}\left(A, \bigcup_{i \in I} B_i\right) &= \bigcup_{i \in I} D_{\mathcal{U}}(A, B_i), \\ E_{I_{\mathcal{U}}}\left(A, \bigcup_{i \in I} B_i\right) &= \bigcap_{i \in I} E_{I_{\mathcal{U}}}(A, B_i), & E_{I_{\mathcal{U}}}\left(\bigcap_{i \in I} A_i, B\right) &= \bigcap_{i \in I} E_{I_{\mathcal{U}}}(A_i, B). \end{aligned}$$

Proof Since \mathcal{U} is left-continuous, the second partial map of $I_{\mathcal{U}}$ is right-continuous and the first one is left-continuous by Proposition 2.5 and thus, the results follow easily from definitions. \square

Proposition 4.4. *Let \mathcal{U} be a conjunctive uninorm and $I_{\mathcal{U}}$ its residual implicator. Let A be a gray scale image and let B be a gray-scale structuring element. Further, let $(A_i)_{i=1}^k$ be a finite family of gray-scale images. Then it holds*

$$\begin{aligned} C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}\left(\bigcup_{i=1}^k A_i, B\right) &\supseteq \bigcup_{i=1}^k C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A_i, B), & O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}\left(\bigcup_{i=1}^k A_i, B\right) &\supseteq \bigcup_{i=1}^k O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A_i, B), \\ C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}\left(\bigcap_{i=1}^k A_i, B\right) &\subseteq \bigcap_{i=1}^k C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A_i, B), & O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}\left(\bigcap_{i=1}^k A_i, B\right) &\subseteq \bigcap_{i=1}^k O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A_i, B). \end{aligned}$$

The translation $T_v(A)$ of a fuzzy set A by $v \in \mathbb{R}^n$ is defined by $T_v(A)(x) = A(x - v)$, the scaling $H_\lambda(A)$ of a fuzzy set A by $\lambda > 0$ is defined by $H_\lambda(A)(x) = A(\frac{1}{\lambda}(x))$. It follows, from [11] (see also [3]) that all four basic fuzzy morphological operators are invariant under translation and scaling.

Proposition 4.5. *Let \mathcal{U} be a conjunctive uninorm and $I_{\mathcal{U}}$ its residual implicator. Let A be a gray-scale image, let B be a gray-scale structuring element and let $v \in \mathbb{R}^n$. Then it holds:*

$$\begin{aligned} D_{\mathcal{U}}(T_v(A), B) &= T_v(D_{\mathcal{U}}(A, B)), & D_{\mathcal{U}}(A, T_v(B)) &= T_{-v}(D_{\mathcal{U}}(A, B)), \\ D_{\mathcal{U}}(T_v(A), T_v(B)) &= D_{\mathcal{U}}(A, B). \end{aligned}$$

The fuzzy erosion $E_{\mathcal{I}_{\mathcal{U}}}(A, B)$ satisfies the same relations. The fuzzy closing $C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}$ holds

$$\begin{aligned} C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(T_v(A), B) &= T_v(C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B)), & C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, T_v(B)) &= C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B), \\ C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(T_v(A), T_v(B)) &= T_v(C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B)), \end{aligned}$$

the fuzzy opening $O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}$ satisfies the same relations than the previous ones.

Proposition 4.6. *Let \mathcal{U} be a conjunctive uninorm and $I_{\mathcal{U}}$ its residual implicator. Let A be a gray-scale image, let B be a gray-scale structuring element and let $\lambda > 0$. Then it holds:*

$$\begin{aligned} D_{\mathcal{U}}(H_\lambda(A), H_\lambda(B)) &= H_\lambda(D_{\mathcal{U}}(A, B)), & C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(H_\lambda(A), H_\lambda(B)) &= H_\lambda(C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B)), \\ E_{\mathcal{I}}(H_\lambda(A), H_\lambda(B)) &= H_\lambda(E_{\mathcal{I}}(A, B)), & O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(H_\lambda(A), H_\lambda(B)) &= H_\lambda(O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B)). \end{aligned}$$

Just as in the binary and the classical gray-scale cases, this property also holds for $\lambda < 0$. For $\lambda = -1$, we obtain the following special case.

Proposition 4.7. *Let \mathcal{U} be a conjunctive uninorm and $I_{\mathcal{U}}$ its residual implicator. Let A be a gray-scale image and let B be a gray-scale structuring element, then it holds*

$$\begin{aligned} -D_{\mathcal{U}}(A, B) &= D_{\mathcal{U}}(-A, -B), & -E_{\mathcal{I}}(A, B) &= E_{\mathcal{I}}(-A, -B), \\ -C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B) &= C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(-A, -B), & -O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B) &= O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(-A, -B). \end{aligned}$$

The principle of local knowledge for the fuzzy dilation and fuzzy erosion is expressed in the following proposition, and is adapted from [3]. We put $d_B = \sup B = \{x \in \mathbb{R}^n \mid B(x) > 0\}$.

Proposition 4.8. *Let \mathcal{U} be a conjunctive uninorm and let $I_{\mathcal{U}}$ be its residual implicator. Let A be a gray-scale image, let B a gray-scale structuring element and let Z be a binary mask. Then it holds:*

$$\begin{aligned} D_{\mathcal{C}}(A \cap Z) \cap E_{\mathcal{I}_{\mathcal{U}}}(Z, d_B) &= D_{\mathcal{C}}(A, B) \cap E_{\mathcal{I}_{\mathcal{U}}}(Z, d_B), \\ E_{\mathcal{I}_{\mathcal{U}}}(A \cap Z) \cap E_{\mathcal{I}_{\mathcal{U}}}(Z, d_B) &= E_{\mathcal{I}_{\mathcal{U}}}(A, B) \cap E_{\mathcal{I}_{\mathcal{U}}}(Z, d_B). \end{aligned}$$

The following results are again similar to those stated in [11], but in this case an adapted proof for uninorms is required and consequently it is included. The extensivity of the fuzzy dilation and the anti-extensivity of the fuzzy erosion is ensured by the next proposition.

Proposition 4.9. *Let \mathcal{U} be a conjunctive uninorm with neutral element $e \in]0, 1[$, let $I_{\mathcal{U}}$ be its residual implicator and let B a gray-scale structuring element such that $B(0) = e$. Then the following inclusions hold:*

$$E_{\mathcal{I}_{\mathcal{U}}}(A, B) \subseteq A \subseteq D_{\mathcal{U}}(A, B).$$

Proof As $\mathcal{I}_{\mathcal{U}}$ is the residual implicator of \mathcal{U} , it satisfies $\mathcal{I}_{\mathcal{U}}(e, x) = x$ for all x . Then

$$\begin{aligned} E_{\mathcal{I}_{\mathcal{U}}}(A, B)(y) &= \inf_x \mathcal{I}_{\mathcal{U}}(B(x - y), A(x)) \leq \mathcal{I}_{\mathcal{U}}(B(0), A(y)) = \mathcal{I}_{\mathcal{U}}(e, A(y)) \\ &= A(y) = \mathcal{U}(e, A(y)) = \mathcal{U}(B(0), A(y)) \leq \sup_x \mathcal{U}(B(x - y), A(x)) \\ &= D_{\mathcal{U}}(A, B)(y). \end{aligned} \quad \square$$

Proposition 4.10. *Let \mathcal{U} be a left-continuous conjunctive uninorm and $I_{\mathcal{U}}$ its residual implicator, let A be a gray-scale image and let B be a gray-scale structuring element, then it holds*

1. *The fuzzy closing $C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}$ is extensive: $A \subseteq C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B)$.*
2. *The fuzzy opening $O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}$ is anti-extensive: $O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B) \subseteq A$.*
3. *The fuzzy closing and the fuzzy opening are idempotent, i.e.:*

$$C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B), B) = C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B), \quad O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B), B) = O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B).$$

Proof The first property follows from definition and taking into account that, by Proposition 2.5, $I_{\mathcal{U}}$ satisfies

$$y \leq \mathcal{I}_{\mathcal{U}}(x, \mathcal{U}(x, y)) \quad \text{for all } (x, y) \in [0, 1]^2.$$

Similarly, 2 can be derived from the fact that, by left-continuity, \mathcal{U} satisfies

$$\mathcal{U}(x, \mathcal{I}_{\mathcal{U}}(x, y)) \leq y.$$

In order to prove 3, we have one inclusion immediately using 1:

$$C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B), B) \supseteq C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B).$$

The other inclusion is a consequence of the anti-extensivity of the fuzzy opening, the definitions of the fuzzy operators and the monotonicity of the fuzzy erosion, that is

$$\begin{aligned} C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B), B) &= E_{\mathcal{I}_{\mathcal{U}}}(D_{\mathcal{U}}(C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B), B), -B) \\ &= E_{\mathcal{I}_{\mathcal{U}}}(D_{\mathcal{U}}(E_{\mathcal{I}_{\mathcal{U}}}(D_{\mathcal{U}}(A, B), -B), B), -B) = E_{\mathcal{I}_{\mathcal{U}}}(O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(D_{\mathcal{U}}(A, B), -B), -B) \\ &\subseteq E_{\mathcal{I}_{\mathcal{U}}}(D_{\mathcal{U}}(A, B), -B) = C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B). \end{aligned}$$

Similarly, it can be prove the idempotency of the fuzzy opening. □

Proposition 4.11. *Let \mathcal{U} be a conjunctive uninorm, with neutral element $e \in]0, 1[$ and let $I_{\mathcal{U}}$ be its residual implicator. Let A be a gray-scale image and let B be a gray-scale structuring element such that there exists $z \in \mathbb{R}^n$ where $B(z) = e$, then it holds*

$$E_{\mathcal{I}_{\mathcal{U}}}(A, B) \subseteq D_{\mathcal{U}}(A, B).$$

Proof Analogous to the proof of Proposition 4.9. \square

Proposition 4.12. *Let \mathcal{U} be a conjunctive uninorm, with neutral element $e \in]0, 1[$ and let $I_{\mathcal{U}}$ be its residual implicator. Let A be a gray-scale image and let B be a gray-scale structuring element such that $B(0) = e$, then it holds*

$$D_{\mathcal{U}}(A, B) \supseteq C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B), \quad D_{\mathcal{U}}(A, B) \supseteq O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B), \quad D_{\mathcal{U}}(A, B) \supseteq O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, -B), \\ E_{\mathcal{I}_{\mathcal{U}}}(A, B) \subseteq O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B), \quad E_{\mathcal{I}_{\mathcal{U}}}(A, B) \subseteq C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B), \quad E_{\mathcal{I}_{\mathcal{U}}}(A, B) \subseteq C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, -B).$$

Proof We only prove the first inclusion since the other follows similarly. For all $y \in \mathbb{R}^n$ we have:

$$C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B)(y) = \inf_x \mathcal{I}_{\mathcal{U}}(B(y - x), \sup_z \mathcal{U}(B(z - x), A(z))) \\ \leq \mathcal{I}_{\mathcal{U}}(B(0), \sup_z \mathcal{U}(B(z - y), A(z))) = \sup_z \mathcal{U}(B(z - y), A(z)) = D_{\mathcal{U}}(A, B)(y). \quad \square$$

Proposition 4.13. *Let \mathcal{U} be a conjunctive uninorm, with neutral element $e \in]0, 1[$ and let $I_{\mathcal{U}}$ be its residual implicator. Let A be a gray scale-image and let B be a gray-scale structuring element such that $B(0) = e$, then it holds that*

$$E_{\mathcal{I}_{\mathcal{U}}}(A, B) \subseteq O_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B) \subseteq A \subseteq C_{\mathcal{U}, \mathcal{I}_{\mathcal{U}}}(A, B) \subseteq D_{\mathcal{U}}(A, B).$$

Proof It is obvious by joining some of the results in the propositions above. \square

Proposition 4.14. *Let \mathcal{U} be a left-continuous conjunctive uninorm, and let A, B and C be gray-scale images. Then it holds that*

$$D_{\mathcal{U}}(A, D_{\mathcal{U}}(B, C)) = D_{\mathcal{U}}(D_{\mathcal{U}}(A, B), -C)$$

and

$$D_{\mathcal{U}}(A, B) = -D_{\mathcal{U}}(B, A).$$

Proof Using left-continuity, it follows easily from the associativity and the commutativity of the uninorm \mathcal{U} . \square

Combinations of dilations and erosions are addressed in the next propositions.

Proposition 4.15. *Let \mathcal{U} be a left-continuous conjunctive uninorm, let $I_{\mathcal{U}}$ be its residual implicator and let A, B and C be gray-scale images. Then it holds:*

$$D_{\mathcal{U}}(D_{\mathcal{U}}(A, B), C) = D_{\mathcal{U}}(D_{\mathcal{U}}(A, C), B), \quad E_{\mathcal{I}_{\mathcal{U}}}(E_{\mathcal{I}_{\mathcal{U}}}(A, B), C) = E_{\mathcal{I}_{\mathcal{U}}}(E_{\mathcal{I}_{\mathcal{U}}}(A, C), B).$$

Proof Using Proposition 4.7 and the previous one we obtain the first equality:

$$D_{\mathcal{U}}(D_{\mathcal{U}}(A, B), C) = D_{\mathcal{U}}(A, D_{\mathcal{U}}(B, -C)) = D_{\mathcal{U}}(A, -D_{\mathcal{U}}(-C, B)) \\ = D_{\mathcal{U}}(A, D_{\mathcal{U}}(C, -B)) = D_{\mathcal{U}}(D_{\mathcal{U}}(A, C), B).$$

To prove the second part, note that since \mathcal{U} is left-continuous, the residual implicator $I_{\mathcal{U}}$ satisfies the exchange principle (see Proposition 2.5) and also, its second partial map is right-continuous. Then, applying definitions and using these facts the results follows. \square

Proposition 4.16. *Let \mathcal{U} be a left-continuous conjunctive uninorm and let $I_{\mathcal{U}}$ be its residual implicator. Then it holds:*

$$E_{\mathcal{I}_{\mathcal{U}}}(A, D_{\mathcal{U}}(B, -C)) = E_{\mathcal{I}_{\mathcal{U}}}(E_{\mathcal{I}_{\mathcal{U}}}(A, B), C).$$

Proof Using Proposition 2.5 and the properties of the implicator $I_{\mathcal{U}}$ we have:

$$\begin{aligned} E_{\mathcal{I}_{\mathcal{U}}}(A, D_{\mathcal{U}}(B, -C))(y) &= \inf_x \mathcal{I}_{\mathcal{U}}(D_{\mathcal{U}}(B, -C)(x - y), A(x)) \\ &= \inf_x \mathcal{I}_{\mathcal{U}}(\sup_z \mathcal{U}(C(-z + x - y), B(z)), A(x)) \\ &= \inf_x \inf_z \mathcal{I}_{\mathcal{U}}(C(-z + x - y), \mathcal{I}_{\mathcal{U}}(B(z), A(x))) \\ &= \inf_{z'} \inf_x \mathcal{I}_{\mathcal{U}}(C(z' - y), \mathcal{I}_{\mathcal{U}}(B(x - z'), A(x))) \\ &= \inf_{z'} \mathcal{I}_{\mathcal{U}}(C(z' - y), \inf_x \mathcal{I}_{\mathcal{U}}(B(x - z'), A(x))) \\ &= \inf_{z'} \mathcal{I}_{\mathcal{U}}(C(z' - y), E_{\mathcal{I}_{\mathcal{U}}}(A, B)(z')) = E_{\mathcal{I}_{\mathcal{U}}}(E_{\mathcal{I}_{\mathcal{U}}}(A, B), C)(y). \quad \square \end{aligned}$$

Although fuzzy dilation and erosion, for a suitable election of the conjunctive and the implicator, are dual, this does not mean that one can freely perform cancellation on fuzzy morphological equalities. In particular, the equality $A = E_{\mathcal{I}}(B, C)$ does not necessarily imply $D_{\mathcal{C}}(A, -C) = B$. However, the following proposition indicates that a containment relationship is still maintained.

Proposition 4.17. *Let \mathcal{U} be a left-continuous conjunctive uninorm and let $I_{\mathcal{U}}$ be its residual implicator. Then it holds:*

$$A \subseteq E_{\mathcal{I}_{\mathcal{U}}}(B, C) \Leftrightarrow D_{\mathcal{U}}(A, -C) \subseteq B.$$

Proof From the definition of $\mathcal{I}_{\mathcal{U}}$ we have that, for all $x, y, z \in [0, 1]$.

$$\mathcal{U}(x, z) \leq y \Leftrightarrow z \leq \mathcal{I}_{\mathcal{U}}(x, y)$$

and, from this equivalence, the result follows trivially. □

5 Conclusions and future work

We have proved that it is possible to use left-continuous conjunctive uninorms in order to construct a fuzzy mathematical morphology satisfying the same properties than in binary and gray-scale classical mathematical morphology. Moreover, we have proved that the properties needed in order to obtain a “good” morphology are satisfied in both cases:

- When we use a representable uninorm \mathcal{U} with additive generator h and the strong negation given by

$$\mathcal{N}(a) = h^{-1}(-h(a)).$$

- When we use the left-continuous, idempotent uninorm $\mathcal{U}^{\mathcal{N}}$ associated to a strong negation \mathcal{N} .

In particular using this kind of uninorms we have duality and idempotence of the fuzzy morphological operators, moreover we have anti-extensivity of the fuzzy opening and extensivity of the fuzzy closing. Intuitive requirements on the effects of the transformations such as expanding, contracting and filtering correspond to these algebraic properties. Namely, extensivity express that a transformation expands a set, whereas anti-extensivity formalizes contracting. Moreover all these properties jointly with increasingness are the mathematical properties necessary for representing filtering. Then, these results lead us to the possibility of deriving fuzzy morphological filters having the same properties as in the classical morphology. On the other hand, from Proposition 4.8 it follows that $D_{\mathcal{U}}(A, B) \setminus E_{\mathcal{U}}(A, B)$ will serve as an edge detector of the image A (as a fuzzy morphological gradient). Also, Proposition 4.4 and Proposition 4.5 shows that the fuzzy morphological operators are invariant under translation and scaling. Invariance under translation implies that the fuzzy morphological operations are independent of the choice of the origin, while invariance under scaling means that these operations are independent of the used scale. Thus, the structuring element only depend on its shape and this can be exploited for directional transformation, which are useful in granulometry.

The next step is to study the effect of the neutral element election in practical cases; that is, the implementation of the fuzzy operators using different uninorms changing its neutral element. The study of closed and open objects, as well as the use, in shape recognition and image analysis, of fuzzy morphological filters based on these fuzzy morphological operators, will be done in future works.

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