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Efficient polygon decomposition into singular and regular regions via Voronoi diagrams

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Abstract. A new polygon decomposition into regular and singular regions is defined; it is a concept that is useful for skeleton extraction and part analysis of elongated shapes. Polygon regions that are narrow according to the Voronoi diagram of the polygon are extended through the boundary that is adjacent and quasiparallel. Regular regions are the narrow ones surrounded by smooth quasiparallel contour segments, while singular regions are the polygon regions that are not regular. We present an efficient algorithm to calculate the decomposition and make a comparative study with previous algorithms.

Keywords: Shape analysis - Skeleton – Corridor

1 Introduction

In the last decade, some researchers have noticed the importance of dividing figures into *regular* and *singular* regions [15,17]. Regular regions correspond to the ones surrounded by smooth noncrossing quasiparallel contour segments. The other regions, the ones on crossings or endings or near branches or nonsmooth contour lines, are the singular regions. Decomposition into regular and singular regions is especially useful for line figures, drawings, characters, etc. [3,20,22], where regular regions are strokes and singular regions are their connections or intersections. In singular regions, it is generally agreed that most skeleton algorithms do not represent the shape well.

This paper was motivated by the need to improve the efficiency of our previous work [16]. The results are a new definition with an algorithm of a lower computational complexity. Our new definition iteratively modifies regions associated with the traditional medial axis transform applying the two criteria. The contribution of this paper is twofold: a better agreement, so we believe, with human perception and a more efficient algorithm to calculate a decomposition based on Voronoi diagrams. We have no knowledge of any other algorithm for polygons that modifies medial axes to get a decomposition into singular and regular regions and that uses no external parameters.

Ogniewicz et al. [13] show how to calculate skeletons that are very stable under noise, but their method uses external parameters. In addition, no matter what parameters are used the skeleton calculated for a perfect rectangle by their method is the typical line with two shorter segments on each side.

Our criteria are the degree of parallelism between two symmetrical segments and the degree of closeness between them, according to the Voronoi diagram. The best segment pairs according to the closeness criterion are matched first. Then the two criteria are used to modify the pairs, improving with each step the degree of parallelism. The representation that we use for a pair of segments that are highly parallel and near each other is a symmetrical trapezoid. *Opposite segments* in a trapezoid are quasiparallel segments that form the regular regions and correspond to the best segment pairs.

Our method can be understood as a modification of the medial axes calculated by the Voronoi diagram so that a parallelism criterion is also used. A medial axis corresponds to a trapezoid bisectrix, and it is considered a "distortion" if it does not seem to respect the parallelism criterion.

Although there are hundreds of thinning algorithms, they either try to correct the distortions after they arise in the process of finding the medial axis transform or they adopt a different method for finding the skeleton. In the first approach, as in References [1,12,18], they have models of distortions but sometimes just a subset of them, based on ad hoc perceptual parameters that use the information preserved by the thin lines. Nevertheless, for most practical purposes the results are very promising. In the second approach, as in References [5,10,4], the process finds a skeleton that avoids the distortions, but no adequate explanation is given to the several criteria used at different stages. Most of them are based on a pixel-by-pixel analysis of the figure.

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In this paper, we present a rather different approach that relies on only two natural criteria with no parameters at any stage; in addition, the approach works only on polygons. If the input is not a polygon but a digital image, a polygonal approximation is needed. Since our algorithm relies on the degree of parallelism of lines, the polygon segments should be good line approximations of the contour; within an application context (resolution level, noise level, figure type, etc.) the polygonal approximation method should have line detection and approximation performances suited to the application context at hand. In short, for the complete process from digital images to parts to work well, the line detection and approximation problems should be solved before the analysis defined here is applied.

The following is a summary of the algorithm:

regular-singular-decomposition
Input: A polygon
Output: Regular and singular regions of a polygon
Calculate the Voronoi diagram
Remove Voronoi edges adjacent to vertices of
    angle $\geq 90°$
Locate Voronoi edge branching points
Remove Voronoi edges adjacent to branching points
    related to vertex segment and vertex-vertex
Split the polygon into corridors and presingular regions
Expand-reduce corridors
end

An example of the whole procedure is shown in Fig. 4. In the following sections, we explain the meaning of each step.

Closeness among the contour segments is measured according to the Voronoi diagram, which is used to find elongated regions (corridors). Voronoi edges (traditional skeleton edges) are the solid base on which our method relies to find the seed of elongated regions. However, we eliminate Voronoi edges that are not robust from the beginning and use other criteria to grow elongated regions. This procedure of growing from each elongated region (expand-reduce corridor process) takes place locally in the intersection or branching (singular) regions; the elongated regions that can be grown in a better quasiparallel fashion are expanded and the other ones are perhaps reduced. For instance, in a rectangle, the elongated region defined by the main axis consumes the other four regions at the corners. Since the most suited elongated regions grow in each iteration, the final result is a set of elongated regions well suited in quasiparallel terms. The algorithm does not reach an optimum with respect to a predefined measure of decomposition quality, but perhaps there is no need for one to carry out pattern analysis and recognition.

2 Perceptual basis

The perceptual criteria on which the method is based are the closeness and parallelism degrees between two symmetrical segments. The first criterion is widely used in all kinds of problems related to pattern recognition and is used in particular to find axes of sets in any dimension. The medial axis transform is one of the 2D methods based on this criterion. The second criterion, parallelism, is well known from the Gestalt school that produces perceptually valid groupings. For instance, for his object-recognition system Lowe [11] makes extensive use of the fact that elements that lie parallel to a common line naturally group together. The main problem is how to combine these two criteria.

Ullman [21] asserts, and has evidence, that among his visual routines is the *bounded activation*. This operation consists of the spread of focus and activation over a surface on the base representation emanating from a given location or contour and stopping at discontinuity boundaries. This routine, together with *boundary tracing*, permits the application of visual routines to regions near others already considered if there are no singularities in between. Region-growing techniques are based on this expansion. We apply this principle to our context in the following way: the focus on an elongated region can be spread over its neighboring area while the opposite contours of the region are quasiparallel.

When a segment can be associated with two different segments, one that is *closer* and another that is *parallel*, we associate the closest ones unless the parallel ones are a *continuation* of a region already considered narrow. In other words, our main perceptual assertion is that narrow parts are perceptually grouped and then spread over their extremes as long as they are quasiparallel. Notice that two perfectly parallel segments that face each other are not necessarily associated unless they are adjacent to a narrow region. If this is the case, they are preferred to the association of one of them with a closer segment.

Our decomposition strategy is specially suited to line figures like characters, line drawings, or other elongated figures, and it is not as general as other shape-recognition principles. However, it has similarities with well-known rules. For instance, the *minima rule* [7] allows the division of a plane curve into parts at negative minima of curvature. This rule breaks the contour at concave vertices; thus narrow regions (necks) are also found in this way for part decomposition.

3 Voronoi diagrams of polygons

Let us assume that we have a polygon with, say, polygonal holes. The polygon is defined by several sequences of oriented segments. In each sequence, called a contour, two segments that follow each other are adjacent, and the first segment is adjacent to the last one. Also, the polygon interior is assumed to be on the left side of the segments. The Voronoi diagram of an arbitrary polygon has been studied by a number of people and can be computed in $O(n \log n)$ time, where n is the number of polygon segments [2]. By their own definitions, it is also true that the Voronoi edges for a polygon contain its medial axis transform, e.g., its skeleton, in the most traditional sense of this term. Therefore, it is not surprising that the Voronoi diagram is useful for defining the intuitive notion of *neighbor segments* that will lead us to our regular region definition.

We will briefly recall some notation of Voronoi diagrams for line segments [9]. As usual, the polygon is supposed to be composed of open segments and vertex points, which are the elements of the polygon. Also, a polygon divides the plane into inside and outside points. If e is an element, the Voronoi region $V(e)$ is the locus of inside points closer to element e than to any other element. Distances to segments are only taken orthogonally to the segment. The boundary edges of a Voronoi region are called *Voronoi edges*. The vertices of the region are called *Voronoi points*. The Voronoi edges shared by the Voronoi regions of two polygon segments are portions of a bisectrix of the lines that contain the segments, while the Voronoi edges shared by a vertex and a segment are portions of a parabola. The Voronoi edges shared by two vertices are also straight lines.

A Voronoi region of a segment s has as boundary s itself and some Voronoi edges. The following observation is simple but very useful.

Proposition 1. *Let s and $V(s)$ be a segment and its Voronoi region. Let P be a point on a Voronoi edge of $V(s)$. The line segment from P to s that is orthogonal to s is contained in $V(s)$, or it is part of a Voronoi edge between $V(s)$ and the Voronoi region of an ending of s.*

Proof: consider the circle C centered in P and tangent to s on a point Q. Since P is on a Voronoi edge, C is completely contained in the polygon; therefore, all polygon segments are outside C. The segment PQ is orthogonal to s because distances to segments are only taken orthogonally. If Q is one of the endings of s, then PQ is part of the edge between $V(s)$ and the Voronoi region of the ending.

Assume that Q is in s. If any point of the segment PQ is outside $V(s)$, then there exists a point R in PQ that is a Voronoi edge of $V(s)$; consider a circle C' centered in R and tangent to s; C' is also tangent at the point Q, and C' is completely contained in the circle C. Therefore, it is impossible for C' to be tangent to another polygon segment, and R cannot be a Voronoi edge. This contradiction implies that R cannot exist. □

Consider a point P on a Voronoi edge between two Voronoi regions associated with two segments. On each Voronoi region, consider the orthogonal projection of P on each segment; when we sweep the complete Voronoi edge by moving P, the orthogonal projections sweep the segments and define two subsegments, one on each segment, that are associated through the Voronoi edge.

Motivated by the previous statement, for each segment s take the orthogonal projection over s of the Voronoi points adjacent to $V(s)$. In the remainder of this paper, we will use the term polygon segments for the subsegments in which the original segments are divided according to the Voronoi diagram. Also, we will work only with the new Voronoi diagram, associated with the new segments, that has some additional edges that divide old Voronoi regions into new ones. As we will see below, segments are naturally partitioned into subsegments according to their closest neighbor segments. For an example, see Fig. 1 in which new edges are added dividing the original contour segments.

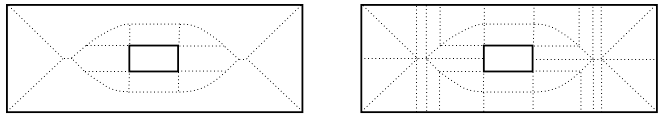

Fig. 1. (*left*) Voronoi edges of a polygon. (*right*) Projections of Voronoi edges over their segments creating subsegments used for defining neighbor segments

Given a segment s, let us consider $V(s)$. According to Proposition 1, the orthogonal projection of a Voronoi point adjacent to $V(s)$ over s is on the interior of s or is one of the vertices of s. However, it cannot be on the interior because polygon segments have already been divided according to these projections. Thus, Voronoi edges are projected on polygon vertices. As a result, the boundary of $V(s)$ is made up of s itself, two Voronoi rectilinear edges orthogonal to s that lie over the vertices of s, and a Voronoi edge e_s that joins these two edges; one of the two orthogonal edges can be reduced to a point. In short, $V(s)$ is bounded by a maximum of four and a minimum of three edges. We have proved the following proposition:

Proposition 2. *Given a segment s, $V(s)$ is bounded by e_s and a maximum of two rectilinear Voronoi edges orthogonal to s.*

Given s, e_s is shared with another Voronoi region. If this region is associated with a segment s', then $e_{s'} = e_s$ and it are a rectilinear edge. Given the distance constrains, $V(s')$ is symmetrical to $V(s)$ with respect to e_s and thus s and s' have the same length.

Given s, if the other Voronoi region is associated with a vertex v, then e_s is a paraboloic edge and v is unique because a parabola has only one vertex.

Definition 1 (Neighbor segments). *Two open polygon segments s_1 and s_2 are* neighbors *if $e_{s_1} = e_{s_2}$. In other words, their Voronoi regions share the Voronoi edge that is the bisectrix of the two segments.*

If two segments are neighbors, we say that one is a neighbor of another and that they are neighbor segments. When two segments are neighbors, there is an interior circle that is tangent to both segments and that does not contain any point of other polygon elements. Notice also that each Voronoi edge has two and only two polygon elements associated with it.

Although in this paper we do not use the concept of *MAT skeleton* of a polygon, the reader should know that all Voronoi edges that are not adjacent to a concave vertex of the polygon are part of the MAT skeleton.

4 Corridors

In this section, we define corridors, polygon regions used to define regular regions.

Definition 2 (Corridor). *A* corridor *is a sequence of $n+2$ adjacent oriented segments $d_1, s_1, \ldots, s_m, d_2, s_{m+1}, \ldots, s_n$, where each s_i is a polygon segment and d_1 and d_2 are segments contained in the interior or boundary of the polygon and called* doors*. The segments form a circular chain, and a door can be reduced to a point. It is assumed that the polygon interior is on the left of all segments.*

In short, a corridor consists of a polygonal chain, a door, another polygonal chain, and the other door that ends where the first chain starts. The doors are, in fact, two (new) segments that join the polygonal chains and are inside the polygon. For example, the synthetic Fig. 6 has three simple corridors with a darker interior, two with four segments (including the doors) and the other one with five segments, one of which is a subsegment of a larger original polygon segment. Figure 4d is more complex: it shows eight corridors in white separated by black regions; there is one large corridor for the top arc of the "A", two corridors for the vertical strokes, one for the horizontal stroke, and four small corridors for the serifs; of these, three have pointed ends; these pointed ends are doors reduced to a point. It is useful to recall that each door in the corridor has two polygon segments adjacent to it within the corridor, a preceding one and a following one.

Corridors and doors in polygons are also used by Kappor et al. [8], and the definitions are compatible. However, the corridors we build below have special properties related to *closeness*.

After finding Voronoi regions and edges of a polygon, we remove Voronoi edges adjacent to concave vertices. We also remove Voronoi edges adjacent to convex vertices that have an angle of 90° or more.

After this removal, the Voronoi points that are adjacent to three or more Voronoi edges are the *branching points*. If we remove the branching points from the skeleton (i.e., if we split the Voronoi edges in the branching points), we obtain sequences of Voronoi edges joined by Voronoi points, each of which has two adjacent Voronoi edges, i.e., of degree two.

Voronoi edge sequences are formed. From each sequence ending the longest subsequence of edges, each of which is either a paraboloid arc (an edge between Voronoi regions for a vertex and a segment) or an edge between Voronoi regions of two vertices, is removed. This is because near branching points segments may have several associations with *opposite* segments, and vertex associations are even less reliable because the parallelism criterion cannot be applied. In this way, we remove edges that we believe do not have enough perceptual evidence to be middle lines of two segments.

Let $e_1, e_2, \ldots, e_k$ be a Voronoi edge sequence and $p_0, \ldots, p_k$ the sequence of its Voronoi points; e_i is, in fact, an oriented edge from p_{i-1} to p_i. Let f_i and f'_i be the two polygon elements associated with e_i.

The new segments d_1 and d_2 will be the *doors* and are defined as follows: d_1 is a segment from the ending point in f_1 corresponding to p_0 to the corresponding ending point in f'_1; d_2 is a segment from the ending point in f_k corresponding to p_k to the corresponding ending point in f'_k.

A corridor wall is defined as follows: consider all segments in the oriented polygon contour sequence starting at the ending point of d_1 on f_1 and finishing at the ending point of d_2 on f_k; in a circular sequence of polygon segments, there are always two ways to go from one point to another: take the subsequence of segments that include f_1 and f_k, which are segments. The other wall is defined analogously using the other door endings.

Finally, the two subsequences and the doors are put together so that the polygon interior is on the left of all segments, and two segments that follow each other in the sequence are adjacent. Notice that the final sequence, including the doors, is circular, in the sense that the last segment is also adjacent to the first door.

These two corridor walls are, in some sense, opposite. More precisely, some (most) of the segments in one chain are neighbors with segments in the other chain.

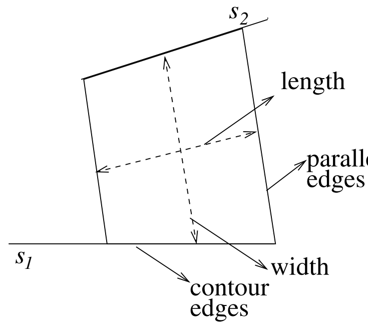

Fig. 2. An example of a regular contour trapezoid over segments s_1 and s_2

5 Trapezoid definitions

An *interior* trapezoid is one that lies completely in the interior of the polygon. We will work only with interior trapezoids. Let us remark that a trapezoid is allowed to be reduced into a triangle.

Definition 3 (Regular and contour trapezoids). *A trapezoid is* regular *if it is symmetrical with respect to an axis perpendicular to its parallel edges. Therefore, it has two edges that are parallel and the other two have equal lengths.*

A contour trapezoid *is a regular trapezoid such that its two edges of equal length lie on the polygon contour. These two edges are called* contour edges*. The other two edges are called* interior *or* parallel edges.

Henceforth all trapezoids are assumed to be interior contour trapezoids. Figures 2 and 4b, 4c, and 4d. show some trapezoids of a polygon. Given a trapezoid, its *bisectrix* is the segment that divides the regular trapezoid into two symmetrical parts. The *trapezoid length* is the length of its bisectrix. The *trapezoid width* is the average length of the parallel sides of the trapezoid.

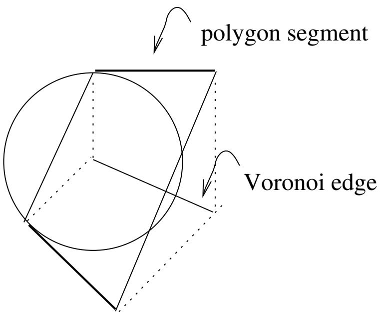

Fig. 3. A neighboring trapezoid

Our basic assumption is that a trapezoid is a building block for regular regions. Its symmetry is a guarantee of contour regularity, and therefore its bisectrix would be a natural part of the skeleton, except when its contour edges are too far apart.

Definition 4 (Degree of parallelism). *The* degree of parallelism *of a trapezoid is the cosine of the angle between the lines on which the trapezoid's contour edges lie.*

Notice that the higher the degree of parallelism, the lower the angle between the contour edges. The only trapezoids allowed are the ones with a degree of parallelism greater or equal to 0, e.g., when the contour edges form a maximum angle of 90°.

Definition 5 (Neighboring trapezoid). *Let this be a trapezoid with contour edges over the polygon segments s_1 and s_2. It is called a* neighboring trapezoid *if s_1 and s_2 are neighbors.*

Figure 3 summarizes the geometric reasons that make a neighboring trapezoid indeed interior and symmetrical: since lines from the Voronoi edge to the segments have equal length and are orthogonal, the trapezoid is regular; the maximal circles centered on the Voronoi edge show that the trapezoid is interior.

The contour edges of a neighboring trapezoid inhere from Voronoi diagrams the perceptual property of closeness, much used in computational perception, in particular by the medial axis transform. However, these traditional skeletons suffer from some drawbacks when the closeness criterion is the only one considered. We will also use a parallelism criterion.

Definition 6 (Compatible trapezoids). *Two trapezoids are* compatible *if their intersection area is void.*

We consider sets of trapezoids that are pairwise compatible. Let us consider two noncompatible trapezoids. There are two cases: either they share a contour segment (or part thereof), or they do not. In the first case, alternative opposite segments for the contour part are suggested, as shown in Fig. 5. In the second case, we say that they *cross* each other.

Notice that two different neighboring trapezoids never cross each other because of basic Voronoi diagram properties. Also, they are always compatible because a segment is a neighbor of one and only one segment.

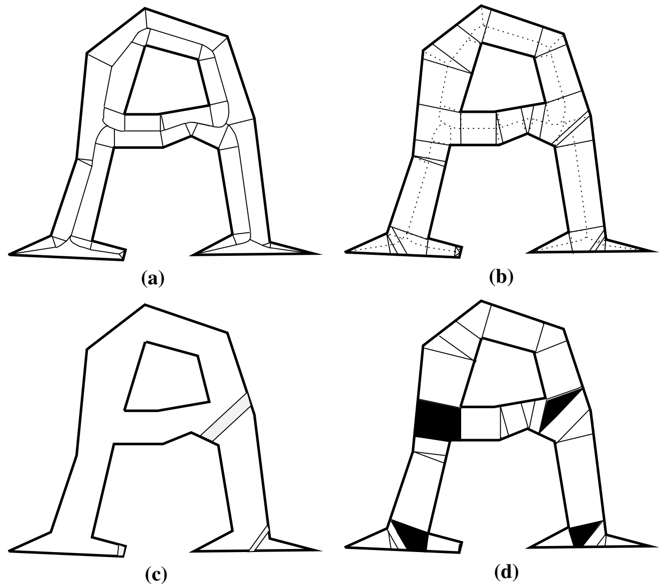

Fig. 4. Construction of a polygon decomposition. **a** The Voronoi diagram. **b** Neighboring trapezoids and original corridors. **c** Trapezoids expanded from corridors. **d** Final result, which shows singular regions in black and final corridors in white

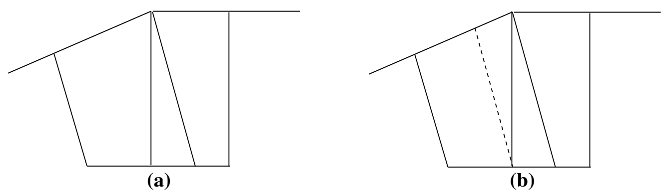

Fig. 5. The trapezoid on the left is divided into two parts, so one is compatible with the other trapezoids

The following definition, illustrated in Fig. 5, takes into account the fact that when a trapezoid t' (on the left in the figure) shares part of the contour with another t, it is possible that a *subtrapezoid* of t' has no relation with the other trapezoid. We take the largest subtrapezoid that is compatible with t.

Definition 7 (Largest subtrapezoid). *Let t' and t be two trapezoids. $sub(t', t)$ is the largest subtrapezoid of t' such that $sub(t', t)$ and t are compatible.*

6 Corridor expansion and reduction

Corridors found from the Voronoi diagram of a polygon by splitting the figure in the branching points represent zones of the polygon where the closeness criterion suggests that the contour parts are opposite each other, i.e., narrow regions are found together with their associated contour chains. The intuitive guiding principle when modifying these regions is the following: the closeness of contours in a corridor is perceptually continued to the adjacent contours, e.g., to the regions nearby, if there is some quasiparallelism in these adjacent contours. In

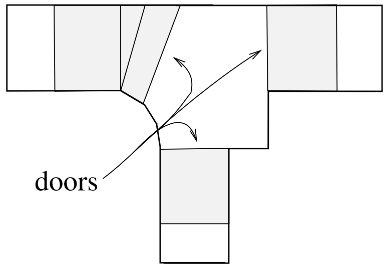

Fig. 6. Corridors are darker than presingular regions. The central presingular region has three doors because it is adjacent to three corridors

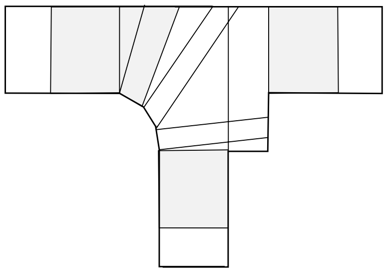

Fig. 7. An adjacent trapezoid to each door of the central presingular region

other words, while expanding a corridor, the principle of "closeness" is relaxed to a principle of "adjacency to a region already considered narrow" while the principle of parallelism is enforced. As the expansion of corridors will cause incompatibilities among them, the parallelism degree is used to compare alternative options at each step. This means that doors will be moved so that corridors will cover less or more area of the polygon but still remain corridors.

6.1 Finding a new trapezoid

Assume we have a set of corridors in a polygon. Consider the complement of the corridor interiors with respect to the polygon interior. A connected component in this complement is called a *presingular* region. Since the only corridor frontiers with the polygon are the doors (because polygon segments are already in the boundary of the polygon), the complement boundary is made up of doors and polygon segments. Let S be a presingular region. The most exterior boundary of S is made up of corridor doors and polygon segment chains (there may be holes within S). In Fig. 6, there are three simple presingular regions, each of which is surrounded by a door and a chain of three contour segments; the other presingular region has three doors and three polygon segment chains.

The next step to get the regular regions from the corridors is the expand-reduce process in which some corridors are expanded, sometimes at the expense of others' reduction. The process is as follows.

Let S be a presingular region with external contour $r_1, \ldots, r_{m_1}, d_1, r_{m_1+1}, \ldots, r_{m_2}, d_2, \ldots, r_{m_k}, d_k,$ where r_j is a polygon segment and d_i is a door. Assume that the polygon interior is on the left of the segments. This sequence is circular so that we can define $d_0 = d_k$. It is possible that $k = 0$, i.e., there are no doors. In this case, this region is a connected component of the polygon that has no skeleton because without corridors there are no parts that have a line shape. What follows is applicable when $k > 0$.

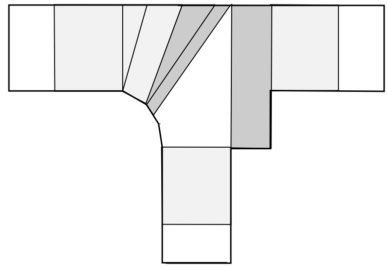

Fig. 8. In the first expansion iteration, the best trapezoid of the three is on the right. In the second step, (part of) the one on the left is selected. The one situated below is never selected because it is incompatible with the one selected in the first step

Consider a door d in a region S and two of its adjacent doors d' and d'' in the contour of S, as shown in Fig. 6.1a: $d', r_1, \ldots, r_m, d, r_{m+1}, \ldots, r_n, d''$. Let C' and C'' be the corridors associated with the doors d' and d'', respectively, and let s' be the previous segment to d' in C' and let s'' be the following segment to d'' in C''.

We are interested in finding a trapezoid such that one of its contour edges is on r_{m+1} and the other is on any $s', r_1, \ldots, r_m$ and that satisfies the following properties. First, the trapezoid must be adjacent to d. Second, the trapezoid must be as long as possible, i.e., cover as much as possible of the segments it lies on. In addition, if the candidate trapezoid, t, is on s', then t must have a higher parallelism degree than the neighboring trapezoid associated with s', within C'.

There is a special case: if r_{m+1} does not exist, i.e., doors d and d'' are adjacent, the trapezoid t could lie on s'' and on any of $s', r_1, \ldots, r_m$. In this case, t must also have a higher degree of parallelism than the neighboring trapezoid associated with s'' in C''. This includes the case of a trapezoid that lies on both s' and s'' and that has a greater parallelism degree than the ones that lie in s' and s''. However, this case is not considered when there is only one door in S.

Analogously, we also consider trapezoids that lie on r_m and one of the segments $r_{m+1}, \ldots, r_n, s''$ with similar properties.

From all the trapezoids that satisfy the above-mentioned properties, we take one with the highest parallelism degree.

For each door d_i we find t_{d_i}, a trapezoid adjacent to d_i under the conditions explained above. And from these we take t_{d_j} with a maximum parallelism degree. This trapezoid is added to the corridor j, and the door d_j is moved to the trapezoid edge further from the corridor j. Thus, one corridor is expanded and another or other corridors may be reduced by moving their S doors until the cor-

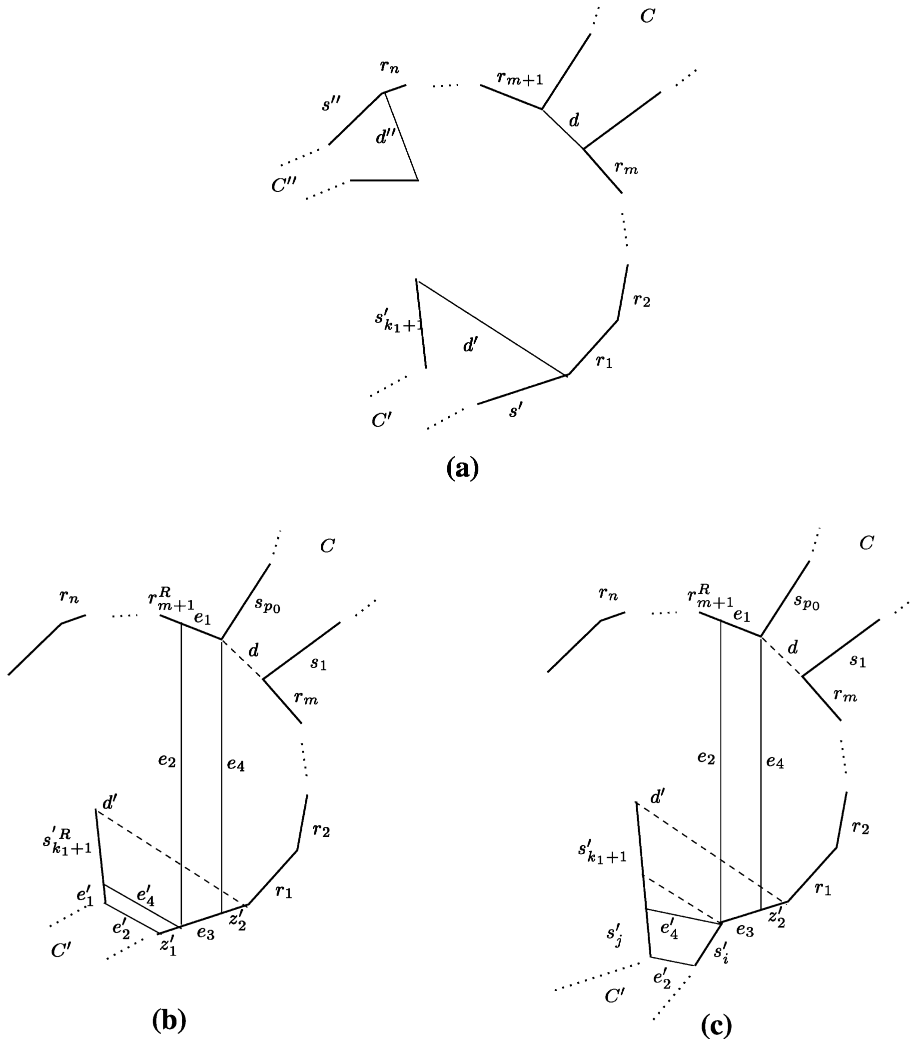

Fig. 9. a Three corridors C, C', and C'' around a singular region. **b** A new trapezoid partially reduces a corridor trapezoid. **c** The corridor trapezoid is completely reduced by the new trapezoid

ridors are compatible, as explained in the following sections. Although the details are technically complex, the driving idea is to expand the corridor associated with d with a trapezoid adjacent to d and with a higher parallelism degree than the others, even the existing ones. To ensure termination when repeating this process, we never allow a corridor that is expanded in one iteration to be considered for reduction in the following iterations.

6.2 Corridor and presingular region updates

Consider a trapezoid t to be added to a corridor C at door d. Assume that the segments around d in S are $d', r_1, \ldots, r_m, d, r_{m+1}, \ldots, r_n, d''$.

There are several cases for t according to the segments it lies on. We discuss the three most representative cases; the others are similar. The cases we explain refer to the situation in which the new trapezoid lies on a segment that belongs to another corridor. The cases are as follows. The first case is when the new trapezoid is partially incompatible with a trapezoid that lies on the other corridor; in this case, the door in this corridor is moved parallel to the old one. The second case is when the new trapezoid is completely incompatible with the trapezoid in the other corridor, and this trapezoid must be removed from the corridor and the next trapezoid in the corridor used to find the new door. And the third case happens when in the second case the corridor has no more trapezoids and the corridor must be removed.

Assume that trapezoid t lies on the segments r_{m+1} and s', where the latter is the segment preceding d' in corridor C', just as before. The trapezoid edges are e_1, e_2, e_3, e_4, where the contour edges are e_1 and e_3 and the segment order leaves the polygon interior on their left. e_1 is part of r_{m+1} and e_3 is part of s'. More precisely, r_{m+1} is made up of e_1 and r_{m+1}^R, and s' is made up of z'_1, e_3, and z'_2. See Fig. 6.1b.

If C is $d, s_1, \ldots, s_{k_0}, d_0, s_{k_0+1}, \ldots, s_{p_0}$, then the new corridor has as a new door segment e_2 and it is $e_2, e_3, z'_2, r_1, \ldots, r_m, s_1, \ldots, s_{k_0}, d_0, s_{k_0+1}, \ldots, s_{p_0}, e_1$.

Assume that t' is a trapezoid that lies on s' within C'; it must be that t has a greater parallelism degree than t' and also that t is as long as possible. Consider $t'_1 = sub(t', t)$, the greatest subtrapezoid of t' compatible with t. The second case in which t'_1 is void is explained later.

If it is not void, we call e'_1, e'_2, z'_1, e'_4 its edges (assuming that the polygon interior is on the left of the edges), and e'_1 and z'_1 are the contour edges. More precisely, s'_{k_1+1} is made up of e'_1 and $s'^R_{k_1+1}$.

If C' is $d'_1, s'_1, \ldots, s'_{k_1-1}, s', d', s'_{k_1+1}, \ldots, s'_{p_1}$, then the new corridor has as a new door segment e'_4 and it is $d'_1, s'_1, \ldots, s'_{k_1-1}, z'_1, e'_4, e'_1, s'_{k_1+2}, \ldots, s'_{p_1}$.

In S, the sequence $d', r_1, \ldots, r_m, d, r_{m+1}, \ldots, r_n, d''$ is replaced by $s'^R_{k_1+1}, e'_4, e_2, r^R_{m+1}, r_{m+2}, \ldots, r_n, d''$, where e'_4 and e_2 are the new doors.

In the case in which no part of t' is compatible with the new trapezoid t, consider s'_i the last segment, if any, in the sequence $s'_1, \ldots, s'_{k_1-1}$ that has a trapezoid, t'_2, associated with edges s'_j, e'_2, s'_i, e'_4 (assuming that the polygon interior is on the left of the edges) for some $j \geq k_1+1$. See Fig. 6.1c. If C' is $d'_1, s'_1, \ldots, s'_{k_1-1}, s', d', s'_{k_1+1}, \ldots, s'_{p_1}$, then the new corridor has as a new door segment e'_4 and it is $d'_1, s'_1, \ldots, s'_i, e'_4, s'_j, \ldots, s'_{p_1}$. In S, the sequence $d', r_1, \ldots, r_m, d, r_{m+1}, \ldots, r_n, d''$ is replaced by $s'_{k_1+1}, \ldots, s'_{j-1}, e'_4, s'_{i+1}, \ldots, s'_{k_1-1}, z'_1, e_2, r^R_{m+1}, \ldots, r_n, d''$, where e'_4 and e_2 are the new doors.

In the last case, if there is no trapezoid that lies on some s'_i and s'_j (t' is the only trapezoid in C'), then corridor C' disappears and must be removed completely by joining S to the presingular region to which the other door d'_1 of C' belongs. The details are analogous to those above.

The other cases depend on the segments the new trapezoid lies on: r_{m+1} and r_i for some $1 \leq i \leq m$, s'' and r_i for some $1 \leq i \leq m$ (i.e., r_{m+1} does not exist and has the same three subcases discussed in this section according to the reduction of the trapezoid that lies on s''), and s'' and s' (which reduces two corridors, each one with the same three subcases). There are also the analogous cases on the other side of d, depending on the segments the new trapezoid lies on: r_m and r_i for some $m+1 \leq i \leq n$ and s' and r_i for some $m+1 \leq i \leq n$ (when r_m does not exist).

6.3 Summary of the process

Each presingular region S is thus modified. This process is repeated until no more corridor modifications are possible. See Fig. 8 for an example of the description of the steps needed to modify a singular region.

There is a special case at the end of the process: when a singular region has two doors and they are adjacent, the two corresponding corridors are merged into one.

Definition 8 (Regular region). *A* regular region *is a corridor after the result of the expand-reduce process.*

Singular regions are the complement of the regular regions in a polygon. The following algorithm summarizes the definitions.

expand-reduce corridors
Input: corridors and presingular regions of a polygon
Output: Regular and singular regions of a polygon.
Sort presingular regions according to their size
For each presingular region S do
  repeat
    let $\{C_i\}_i$, k corridors adjacent to S
    let $\{d_i\}_i$, k doors in S
    Find $\{t_i\}_i$, new trapezoids adjacent to each door
    Let t_j the one with the greatest parallelism degree
    Expand C_j with t_j and move d_j
    For each C_i not compatible with t_j
      Reduce C_i
      If C_i disappears
        merge S to the presingular region on
          the other side of C_i
  until there is no modification in S
end

In the algorithm above, the size of the presingular region is used so that the result does not depend on the application order. We give preference to the bigger regions so that perceptually larger areas have more importance and are decided first. The size of a presingular region can be measured by its perimeter; its area is not used because the region may be narrow but be the connection of large regions.

As we said before, to ensure termination, we never allow a corridor that is expanded in one iteration to be considered for reduction later during the processing of a singular region.

A final comment on skeletons: for regular regions we can take as skeletons the trapezoid bisectrices and their rectilinear extrapolations . Singular regions have no skeleton; they are connection regions (nodes) for regular axes (edges), so connectivity is maintained. In addition, a global shape analysis using *good continuation* and other perceptual criteria, as in Suzuki and Mori [20], can be used to generate associations between regular regions adjacent to a singular region or to remove short regular regions (with their width larger than their length) to improve crossings. In short, for skeletonization purposes, skeletons of regular regions near crossings, corners, etc. are elongated or removed. We think that such a process is application dependent, and it is not discussed here any further.

7 Implementation and complexity analysis

A Voronoi diagram allows us to organize spatially the polygon segments (or any object) in a plane. It will be used not only to find neighboring trapezoids but also to efficiently find candidates for quasiparallel segments for a segment.

We construct the Voronoi diagram for the polygon, which allows us to find the neighboring segments. Orthogonal projection of Voronoi points (Voronoi region vertices) over the contour of Voronoi regions divides the segments into more convenient subsegments. It is well known that the number of Voronoi points is $O(n)$, where n is the number of polygon segments [2]. Each subsegment of a segment s has a natural and unique Voronoi

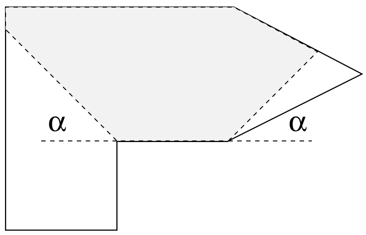

Fig. 10. The α-span region of the lower horizontal segment

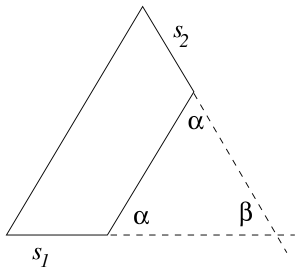

Fig. 11. If β is the angle between the contour segments of a trapezoid, s_1 and s_2, s_1 is within the $Span_\alpha(s_2)$

subregion associated with it, i.e., the set of points of the Voronoi region of s whose orthogonal projection is over the subsegment. Hence there are $O(n)$ subsegments and $O(n)$ Voronoi subregions. As we said earlier, we assume that these subsegments are the original segments of the polygon with their corresponding Voronoi regions. For each Voronoi edge segment b between two polygon segments s_1 and s_2 we construct the trapezoid t that has as contour edges the orthogonal projections of b over s_1 and s_2.

During the expansion process, the contour segments most parallel to a given contour segment are found within a local polygon region. The main problem to be tackled is the visibility between two segments within the polygon. Although there are efficient algorithms to calculate segment visibility [19], we need to analyze it only in a local neighborhood of branching skeleton points. To do so, we calculate the following spanning region for a segment as shown in Fig. 10.

Definition 9 (α-span). *Let s be a polygon segment. Assume that the polygon interior is on the left of s. Consider two rays that start from the vertices of s and form α degrees and $180 - \alpha$ degrees with the segment, respectively, on its left side. The intersection of the polygon (interior and contour) with the region between the rays and the segment that is also connected to s is called the* α-span *of s, $Span_\alpha(s)$.*

We only allow symmetrical trapezoids whose contour edges form an angle less than or equal to 90°. Due to this fact, the interior edges form an angle greater than or equal to 45° with the contour edges. Therefore, all symmetrical trapezoids one of whose edges is on a segment s are contained within $Span_{45}(s)$. Given a contour segment, during corridor expansion we need to find a trapezoid with an angle between its contour edges lower than a given value β.

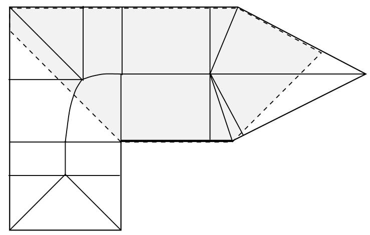

Fig. 12. Only the Voronoi regions that intersect the span should be searched

If the angle between the segments must be lower than β, it is enough to search for a better trapezoid in $Span_\alpha(s)$ with $\alpha = \frac{180-\beta}{2}$, where s is any of the segments of t; the reason behind this value is shown in Fig. 11. In most cases, β is much less than 90°, so the actual computational cost is low. To find the segments that intersect $Span_\alpha(s)$, we search the Voronoi diagram starting with the Voronoi regions adjacent to $V(s)$, the Voronoi region of s. Then, for those regions $V(s')$ that intersect $Span_\alpha(s)$ we search the Voronoi regions adjacent to $V(s')$, and so on. See Fig. 12 for an example of the regions that would be checked.

We use a data structure that has the information of the *dual graph* $D(P)$ of the Voronoi diagram $V(P)$ of the polygon P. The dual graph of a Voronoi diagram has nodes for elements, and two nodes are connected if their associated Voronoi regions share a Voronoi edge. There is a node for each open segment and a node for each vertex. On the dual graph $D(P)$, the search described above is a breadth-first search starting with the node for $V(s)$, and that expands only the nodes of $D(P)$ that intersect $Span_\alpha(s)$.

As was mentioned above, the expansion process is applied within a local neighborhood of a presingular region, which is in fact the presingular region together with a maximum of one neighboring trapezoid from each adjacent corridor. Therefore, the expansion procedure is expected to be very fast for each presingular region. It should not be forgotten that if α is near 90°, the number of trapezoids in the span is small. This is experimentally confirmed, as discussed in the next section.

At this point, we suggest that the reader review the algorithm summary given in the introduction and the example of the whole procedure shown in Fig. 4.

The previously known algorithm for polygon decomposition into singular and regular regions is $O(n^3)$ [16], although for a different definition of regularity with similar results. In this new algorithm, all steps can be carried out with a complexity lower than or equal to $O(n_d n^2)$, where n_d is the maximum number of doors that a presingular region has: assume that a presingular region has a maximum of n_S contour segments. Since for each door a corridor trapezoid is considered, there are $O(n_S + n_d)$ elements in the region and the same order of Voronoi regions. For each door at most this number of regions must be visited to find an adjacent trapezoid, so the complexity is $O(n_d(n_S + n_d))$. This is the worst-case complexity for each iteration in a presingular region. The total number

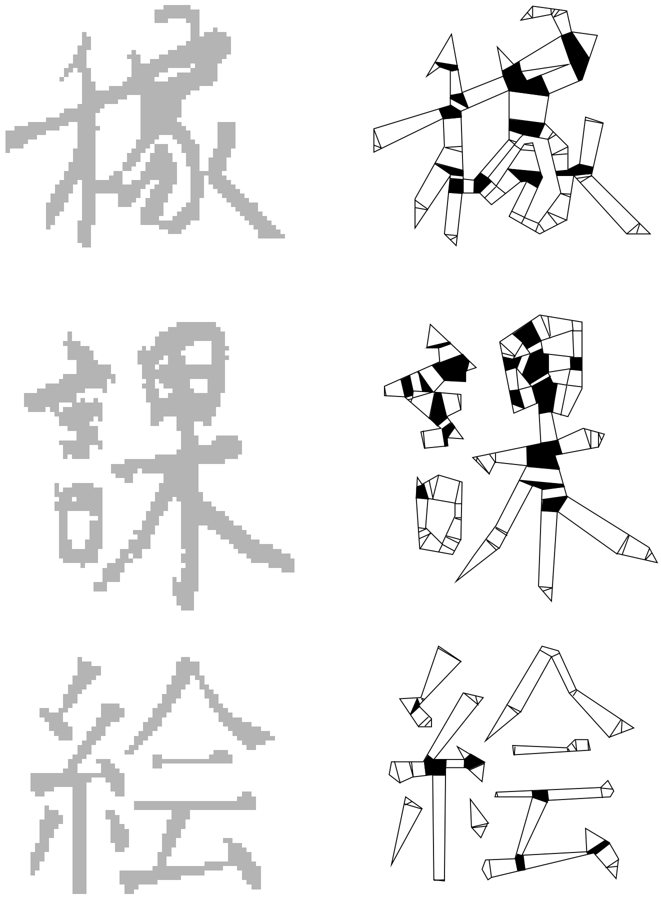

Fig. 13. Polygon decompositions of some Kanji characters. Singular regions are in dark

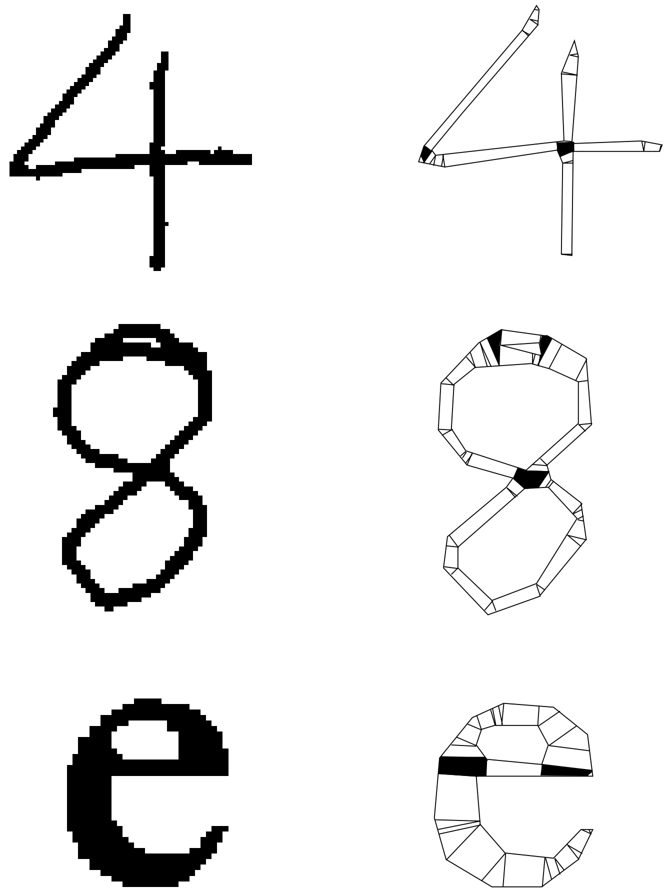

Fig. 14. Polygon decompositions of some Latin characters

of iterations over all presingular regions is bounded by the total number of segments n because a trapezoid that is expanded is never removed. Therefore, the complexity is in the worst case $O(n \times n_d(n_S + n_d))$, and applying $n_S \leq n$ we obtain $O(n_d n^2)$. In our shape database, n_d never exceeded the value of 6 because this number is related to the "branching degree" of a shape, the maximum number of strokes that meet in a region, and it is not natural and physically difficult to draw a large number of strokes and leave an intersection region. Although it is possible to build figures with a large value for n_d, we can say that for all figures with a small branching degree, which are all practical ones, the algorithm has a worst-case quadratic complexity.

8 Experiments

We show several examples of the new algorithm application on letters in Figs. 13 and 14. Our analysis requires a polygonal approximation method (we use one described in Pavlidis [14]) for pixel contours. Black regions are singular regions. Regular regions are sequences of trapezoids.

The algorithm takes 1 s per image on average on a Pentium at 199 MHz. For simplicity, we use a slow but easy implementation of the Voronoi diagram; for a new excellent implementation the reader is referred to Held [6]. After the diagram is calculated, the expand-reduce process takes less than 1% of the time.

The decomposition naturally finds elongated regions in shapes. These correspond to the regular regions that have two clearly defined endings. Singular regions are connection regions of the regular regions, so all together there is no loss of connectivity.

We also tested the algorithm on other types of shapes of real images shown in Fig. 15. The shape at the top of the second column has a hole; singular regions connect only horizontal regular regions. The shape at the bottom of the first column has a big singular region because opposite segments cannot have more than 90°. Our algorithm is not suited to these kinds of round shapes. Notice that trapezoid bisectrices are good skeletons for the elongated shapes.

9 Comparative studies

In Fig. 16, neighboring trapezoids (including the ones that relate segments that be more than 90° between them) are shown. Remember that their bisectrices are all part of the medial axis of each shape. Here we can easily compare the difference of our method with others

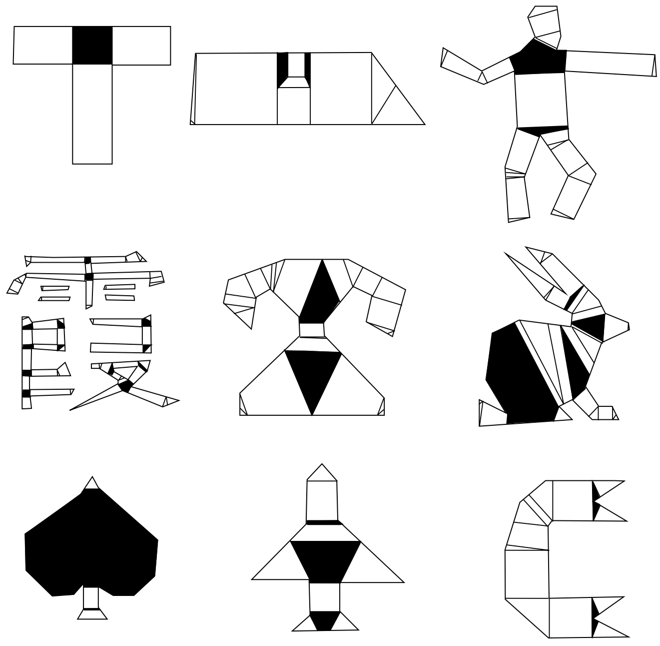

Fig. 15. Polygon decompositions; singular regions in dark

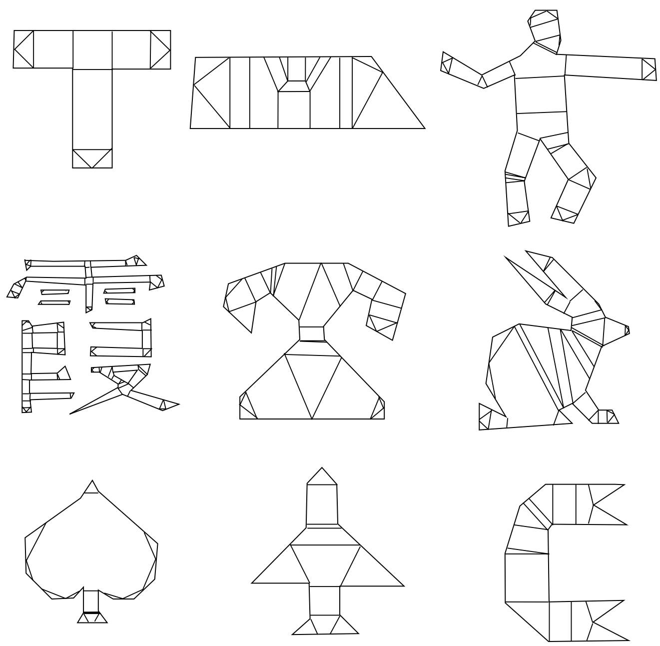

Fig. 16. Polygon decompositions using Voronoi regions only

based on the medial axis transform, especially near stroke endings. Our method, as with other regular-singular analysis methods, is simply better because it avoids the extra skeleton branches near endings.

Zou and Yan [22] have recently developed a method of skeleton extraction of digital images based on regular-singular analysis. It is based on Delaunay triangulations so that their geometric foundation is similar to ours. We see it as a formal approach to Suzuki and Mori's [20] idea on pixel-based images. Zou and Yan's method divides the figure into regular and singular regions and gives similar results to ours, and we consider their work and ours very close, despite the fact that their creation and analysis of singular regions is very different from ours.

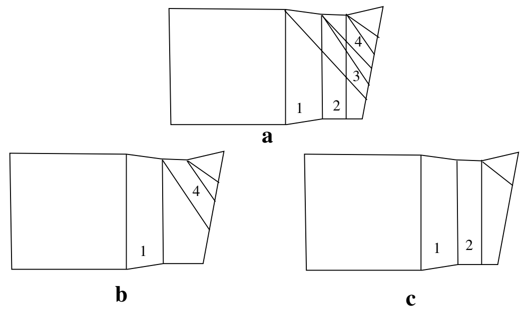

Fig. 17. **a** Some trapezoids. **b** Trapezoid sequence using the old definition: the longest tilted trapezoid, 3, is preferred to the vertical one, 2, so less intuitive segments are associated. **c** Trapezoid sequence using the new definition: the vertical trapezoid, 2, appears in the expansion step

Our method is faster, which is not surprising since theirs is pixel based: they report a computational time on the order of tens of seconds, while our time is on the order of seconds for the same type of figures (characters and a human silhouette) on similar machines (Pentium under 200 MHz).

However, our analysis requires a polygonal approximation method for pixel contours, but it consumes less than 5% of the total time, even for big images (1000×1200 pixels). Therefore, we could say that our method is almost independent of the image size, as expected. Our method depends on the number of edges used in the approximation, which is a small number compared to the number of pixels. Nevertheless, polygonal approximation methods require parameters that are related to the approximation level and are application dependent. Another issue is that although the approximation can erase important features, it will also remove noise to which pixel-based methods are more sensitive.

Our method cannot be applied to pixel-based images because the parallelism criterion cannot be used on segments that are very short; as explained above, lines should be detected before quasiparallel ones are found. Zou and Yan's method cannot be applied to polygons because constraint Delaunay triangulations cannot express correctly local symmetry when edges are long. Therefore, we believe that these two methods solve the same problem in different contexts. However, we have had some results on an algorithm that works with polygons or both small and long edges and it will be explained elsewhere.

Let us now compare our new method to our previous work [16]. Figure 17 shows a shape for which the definition proposed here gives a more perceptually intuitive segmentation than the previous one.

The old definition always gives more importance to the closeness criterion: among crossing trapezoids the narrower wins, and then the parallelism criterion is applied. Thus the tilted trapezoid 3 in Fig. 17 is preferred to the vertical trapezoid 2 (because it is narrower), but

Fig. 18. a Neighboring trapezoids using the old definition. **b** Neighboring trapezoids using the new definition: faraway parallel edges are not associated

trapezoid 2 is discarded later by the other vertical trapezoid 1 that shares a side with it and has more parallel sides. However, the discarded vertical trapezoid 2 on the right is better than the smaller tilted trapezoid 4. Using the new definition, in the first expansion step the vertical trapezoid 2 is preferred, and the larger tilted trapezoid is not compatible and never selected. In the new version presented in this paper, a trapezoid can be "aided" by its context to win over others, even though by itself it would lose. In these kinds of shapes, in which a narrow region can grow in a direction with a good parallelism, the new version performs better. However, we should say that for elongated shapes, in general, the results are very similar. A drawback of the old algorithm still remains: it does not work well when polygon vertices are the figure pixels.

Figure 18 shows another example in which the new definition is better: the old definition associates parallel edges, no matter how far apart they are, as long as there are no other narrower trapezoids crossing in between. The new definition does not match the faraway segments because they are not neighbors or adjacent to neighbor segments.

10 Conclusion

In this paper, we have defined a perceptually guided way to combine a closeness and a parallelism criterion to find regular and singular regions of polygons for elongated shapes. We have also explained a quadratic time algorithm in the number of segments to calculate regular and singular regions on all (practical) polygons, the most efficient to our knowledge, and we have shown several representative examples on real characters and shapes.

The method can be viewed as a modification of Voronoi skeletons of polygons to render them without "artifacts." Its most important features are the independence from external parameters and the use of only two simple criteria. We have shown that our method gives better results than methods based on the MAT and than our previous method, and that it is faster than other methods that give similar results.

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