Introduction

- The objective is the design of state feedback control laws that yield desirable closed-loop performance in terms of both transient and steady-state response characteristics.
- If the open-loop state equation is controllable, then an arbitrary closed-loop eigenvalue placement via state-space feedback can be achieved.
- Some more topics that will be studied include:
  - Explicit feedback gain formulas for eigenvalue placement in the single-input case
  - Relationship between eigenvalue location of linear state equation and its dynamic response characteristics
  - Steady-state performance improvement (integral error compensation)
The open-loop system under study (the plant) is represented by the LTI state equation:
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
\]
(null direct matrix D is assumed)
We focus on the resulting effect of state feedback control laws like:
\[
u(t) = -Kx(t) + r(t)
\]
where K is the constant state feedback gain matrix \((m \times n)\) that yield the closed-loop state equation with the desired performance characteristics
\[
\begin{align*}
\dot{x}(t) &= (A - BK)x(t) + Br(t) \\
y(t) &= Cx(t)
\end{align*}
\]
For a SISO system, K is a \(1 \times n\) row vector and \(r(t)\) a scalar signal, thus, the state feedback control law can be written as:
\[
u(t) = -k_1 x_1(t) - k_2 x_2(t) - \cdots - k_n x_n(t) + r(t)
\]
Dynamic Response Shaping

- In addition to closed-loop stability, the designer is often interested in other characteristics of the closed-loop transient response, such as rise time \( t_r \), peak time \( t_p \), percent overshoot \( M_o \), and settling time \( t_s \) of the step response.
- Specifying desired closed-loop system behavior via eigenvalue selection is called *shaping* the dynamic response (or pole placement).
- First-order and second-order dominant systems are frequently used as approximations in the design process.
- A short resume on how to proceed and basic methods for SISO systems can be found in:
  
  http://dmi.uib.es/goliver/RA/Teoria/5_DomTemporal.pdf

Closed-loop Eigenvalue Placement

The following process is possible thanks to the existence of a connection between the arbitrary choice of the state feedback gain matrix \( K \) (placing the closed-loop eigenvalues) and controllability of the open-loop state equation (matrix \( A \) and \( B \) of the system). This result yields thanks to the next theorem:

- **For any symmetric set of \( n \) complex numbers \( \{\mu_1, \mu_2, ..., \mu_n\} \), there exists a state feedback gain matrix \( K \) such that \( \sigma(A-BK)=\{\mu_1, \mu_2, ..., \mu_n\} \) if and only if the pair \( (A, B) \) is controllable.**

\( \sigma(M) \) denotes the set of eigenvalues of \( M \)

Next, techniques to determine the \( K \) matrix elements for the special case of controllable single-input state equation systems are exposed:

- Feedback Gain Formula for Controller Canonical Form
- Ackermann’s Formula
Closed-loop Eigenvalue Placement

Feedback Gain Formula for Controller Canonical Form (CCF)

The coefficient matrices for CCF are given below:

\[
A_{\text{CCF}} = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
-a_0 & -a_1 & -a_2 & \ldots & -a_{n-1}
\end{bmatrix},
B_{\text{CCF}} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

its characteristic polynomial is written as:

\[sI - A_{\text{CCF}} = s^n + a_{n-1}s^{n-1} + \cdots + a_2s^2 + a_1s + a_0\]

For the single-input case, the gain matrix \(K\) is reduced to a feedback gain vector denoted by:

\[K_{\text{CCF}} = [k_0, k_1, \ldots, k_{n-1}]\]

Thus, the closed-loop system dynamics matrix is:

\[
A_{\text{CCF}} - B_{\text{CCF}}K_{\text{CCF}} = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
-a_0 -k_0 & -a_1 -k_1 & -a_2 -k_2 & \ldots & -a_{n-1} -k_{n-1}
\end{bmatrix}
\]

which characteristic polynomial is:

\[
[sI - A_{\text{CCF}} + B_{\text{CCF}}K_{\text{CCF}}] = s^n + (a_{n-1} + k_{n-1})s^{n-1} + \cdots + (a_2 + k_2)s^2 + (a_1 + k_1)s + (a_0 + k_0)
\]

Now, suppose a set of arbitrary symmetric complex numbers: \(\{\mu_1, \mu_2, \ldots, \mu_n\}\) that give the desired closed-loop behavior. Thus, the associated closed-loop characteristic polynomial is defined by:

\[
\alpha(s) = (s - \mu_1)(s - \mu_2) \cdots (s - \mu_n) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_2s^2 + \alpha_1s + \alpha_0
\]

Thus, while the problem is to determine \(K_{\text{CCF}}\) so that the characteristic polynomial of \(A_{\text{CCF}} - B_{\text{CCF}}K_{\text{CCF}}\) matches the desired closed-loop characteristic polynomial \(\alpha(s)\), we compare the two previous polynomials and equate them term to term obtaining:

\[
\alpha_0 = a_0 + k_0, \quad \alpha_1 = a_1 + k_1, \quad \alpha_2 = a_2 + k_2, \ldots, \alpha_{n-1} = a_{n-1} + k_{n-1}
\]

and the state feedback gain vector is given by:

\[K_{\text{CCF}} = [(\alpha_0 - a_0), (\alpha_1 - a_1), \ldots, (\alpha_{n-1} - a_{n-1})]\]
Exercise 4.1: Given the three-dimensional state equation specified in CCF by:

\[
A_{CCF} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -5 & -3 \end{bmatrix}, \quad B_{CCF} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_{CCF} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\]

by inspection, the open-loop characteristic polynomial is easy to obtain:

\[
a(s) = s^3 + a_2 s^2 + a_1 s + a_0 = s^3 + 3s^2 + 5s + 12
\]

The eigenvalues for this equation are: \(\lambda_{1,2,3} = -2.76, -0.12 \pm 2.08j\).

This system has a typical third-order lightly damped and asymptotically stable step response that can easily be obtained with Matlab.

Suppose that we want to design a state feedback control law to improve the transient response performance. Our objective is:

- \(M_o < 10\%\); \(t_s < 5s\)
- It is easy to obtain that

\[
A_{CCF} - B_{CCF} K_{CCF} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7.46 & -9.36 & -7.68 \end{bmatrix}
\]

Next figure compares the OLoop and CLoop responses to a unit step input.

Note that both results have an important steady-state error. Methods to correct this error will be addressed later.
Closed-loop Eigenvalue Placement

Note that the steady-state value for the previous systems are:
\[
y_{ss} = \lim_{s \to \infty} y(t) = \lim_{s \to 0} sY(s)
\]

Open-loop system response

\[
y_{ss} = \lim_{s \to 0} \frac{1}{s^3 + 3s^2 + 5s + 12} = \frac{1}{12} = 0.083
\]

Closed-loop system response

\[
y_{ss} = \frac{1}{7.46} = 0.134
\]

► Ackermann’s formula: Given a controllable system defined by matrices \( A \), \( B \), \( C \) and \( D \), and a set of desired closed-loop eigenvalues \( \{\mu_1, \mu_2, \ldots, \mu_n\} \), with associated closed-loop characteristic polynomial:
\[
\alpha(s) = (s - \mu_1)(s - \mu_2)\cdots(s - \mu_n) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_2s^2 + \alpha_1s + \alpha_0
\]
the feedback gain vector \( K \) can be obtained as:
\[
K = [0 \ 0 \ \cdots \ 1]P^{-1}\alpha(A)
\]

Where \( P \) is the controllability matrix for the controllable pair \( (A, B) \) and \( \alpha(A) \) is defined as:
\[
\alpha(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_2A^2 + \alpha_1A + \alpha_0I
\]

Thus, \( K \) is function of \( A \) and \( B \), and can be obtained using the Ackermann’s formula, no matter which coordinate representation is used.

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Closed-loop Eigenvalue Placement

► Exercise 4.2: Obtain the gain vector \( K \) for the same system described in the previous example, with the same desired eigenvalues placement

\[
A_{\text{CCF}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -5 & -3 \end{bmatrix} \quad B_{\text{CCF}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C_{\text{CCF}} = [1 \ 0 \ 0]
\]

Matlab code 1

```matlab
>> A=[0 1 0; 0 0 1; -12 -5 -3];
>> B=[0; 0; 1];
>> ALFA=A^3+7.68*A^2+9.36*A+7.46*eye(3);
>> P=ctrb(A,B);
>> K=[0 0 1]*inv(P)*ALFA;
K =
   -4.53   4.36   4.68
```

Matlab code 2

```matlab
>> A=[0 1 0; 0 0 1; -12 -5 -3];
>> B=[0; 0; 1];
>> Pols=[-6.4 -0.64-0.87j -0.64+0.87j];
>> K=acker(A, B, Pols)
K =
   -4.53   4.36   4.68
```

acker() Matlab function is just valid for SISO systems. For MIMO, it should be used place()
Steady-State Tracking

Up to now, we have focused on how state feedback control laws influence the transient response characteristics of a system. All the efforts have been centered on making the most of the freedom to specify closed-loop eigenvalues for a controllable state equation and how eigenvalues locations affect the transient response.

It has been shown that tuning the gain matrix $K$, only some of the transient parameters can be conveniently adjusted, but there is no control on the steady-state value of the system.

We now address the steady-state tracking requirement for step reference inputs. Such control systems are commonly referred to as servomechanism. Two approaches are described:
- **Input gain**: Addition of an input gain to the state feedback control law.
- **Integral action**: Inclusion of an integral action on the tracking error.

**Input Gain**

Consider a new state feedback law of the form:

$$u(t) = -Kx(t) + Gr(t)$$

The resulting closed-loop state equation is:

$$\dot{x}(t) = (A - BK)x(t) + BGr(t)$$

$$y(t) = Cx(t)$$

The reference input $r(t)$ is now multiplied by a gain $G$ to be chosen so that for a step reference input $r(t) = R$, $t \geq 0$, the steady-state of the output is $R$:

$$y_{ss} = \lim_{t \to \infty} y(t) = R$$

To obtain an expression for $G$, we proceed as follows:
- For the constant reference input $r(t) = R$, $t \geq 0$, steady-state corresponds to an equilibrium condition for the closed-loop state equation involving an equilibrium state denoted by $x_{ss}$. Thus, the state equation satisfies:

$$\dot{x}(t) = 0 \Rightarrow (A - BK)x_{ss} + BGR = 0$$

- The steady-state output is obtained from:

$$y_{ss} = Cx_{ss} = -C(A - BK)^{-1}BGR$$

- Thanks to the stated limit condition:

$$R = -C(A - BK)^{-1}BGR \Rightarrow G = -[C(A - BK)^{-1}B]^{-1}$$
Steady-State Tracking

This result can be generalized to address steady-state closed-loop gain other than the unit (identity matrix). If $K_{dc}$ is the desired closed-loop dc gain, the new expression for the input gain to achieve the desired result is:

$$G = -\left[C(A - BK)^{-1}B\right]^T K_{dc}$$

These results are valid for any multiple-input, multiple-output system, provided the open-loop state equation has at least as many inputs as outputs. The SISO case meets these requirements.

**Exercise 4.3**: Modify the state feedback control law computed for the state equation in the previous example to include an input gain chosen so that the OLoop and CLoop unit step responses reach the same steady-state value. Remember that the OLoop state equation in that example was specified by the matrices:

$$A_{CCF} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -5 & -3 \end{bmatrix}, \quad B_{CCF} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_{CCF} = [1 \ 0 \ 0]$$

The step response of the open-loop and closed-loop systems are shown below, both having the same steady-state result, thanks to the input gain adjustment.
Steady-State Tracking

The method previously described requires accurate knowledge of the open-loop state equation’s coefficient matrices in order to obtain \( G = -[C(A-BK)]^{-1}B \). In real situations, there are many aspects (model uncertainty, parameter variations, approximations,…) that result in deviations between the nominal coefficient matrices and the actual system. Thus, a significant difference between the actual and the estimated steady-state behavior can arise. Hence, more robust methods to design servomechanisms that can deal with system parameters uncertainties are needed. Some solutions have been presented in the classical literature to deal with that problem.

**Integral Action**

Adding an integral term to the control law guarantees obtaining a system that yields zero steady-state tracking error for step reference inputs, as long as closed-loop stability is maintained. The next assumptions are stated for the open-loop state equation:

1. \((A, B)\) is controllable.
2. It does not have poles at \( s=0 \) (the integral action wouldn’t be necessary).
3. It does not have zeros at \( s=0 \) (the integral action will not be cancelled).

Moreover, for simplicity, a SISO system is assumed.

The new control law proposed is:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Kx(t) \\
\xi(t) &= r(t) - y(t) \\
u(t) &= -Kx(t) + k_i \xi(t)
\end{align*}
\]

where: \( \xi(t) = r(t) - y(t) \) in which \( r(t) \) is the input (step) reference to be tracked by \( y(t) \). Thus, \( \xi(t) \) represents the integral of the tracking error.

The Laplace transform of \( \xi(t) \) with zero initial conditions gives:

\[
\mathcal{L}\{\xi(t)\} = \frac{E(s)}{s}
\]

meaning that the integral error term introduces an open-loop pole at the origin and, if the stated 2 & 3 assumptions are verified, a type I system is guaranteed. Thus a null tracking error for a step input signal is also ensured.
Steady-State Tracking

Notice that the control law can be written as:

\[ u(t) = -[K - k_*] \cdot \begin{bmatrix} x(t) \\ \dot{\xi}(t) \end{bmatrix} \]

which can be interpreted as a state feedback control law involving an \((n+1)\) dimensional augmented state vector formed by the open-loop state vector \(x(t)\) and the integrator state variable \(\dot{\xi}(t)\). Thus, the new \((n+1)\) dimensional closed-loop state equation is:

\[
\begin{bmatrix}
    x(t) \\
    \dot{\xi}(t)
\end{bmatrix} =
\begin{bmatrix}
    A - BK & Bk_* \\
    -C & 0
\end{bmatrix}
\begin{bmatrix}
    x(t) \\
    \dot{\xi}(t)
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    1
\end{bmatrix} r(t)
\]

meaning that the dynamics (transient response and stability) of the closed-loop system is determined by the eigenvalues of the \((n+1)\)x\((n+1)\) closed-loop system dynamics matrix:

\[
\begin{bmatrix}
    A - BK & Bk_* \\
    -C & 0
\end{bmatrix}
= \begin{bmatrix}
    A & 0 \\
    -C & 0
\end{bmatrix}
- \begin{bmatrix}
    0 & B \\
    0 & 0
\end{bmatrix} [K - k_*]
\]

It can be proved that the arbitrary placement of the closed-loop eigenvalues is guaranteed if the three initial assumptions are fulfilled, which results equivalent to require:

\[
\begin{bmatrix}
    A & 0 \\
    -C & 0
\end{bmatrix}
\begin{bmatrix}
    B \\
    0
\end{bmatrix} = (A^*, B^*)
\]

to be a controllable pair. As a consequence, the \((n+1)\) closed-loop eigenvalues can be arbitrarily placed by appropriate choice of the augmented feedback gain vector \(K^*=[K - k_*]\) in a similar way as it was described for the feedback gain method.

The steps that should be followed to completely design the integral action controlled system are:

- Verify the three initial conditions: \((A, B)\) controllable, no pole and no zero at \(s=0\) for the open-loop system.
- Assign the desired \(n+1\) poles of the system. Different criteria could be followed to that end: dominant 2\(^\text{nd}\) order system, ITAE, …
- Obtain the expanded matrices \(A^*\) and \(B^*\) of the \(n+1\) order system.
  \[
  A^* = \begin{bmatrix}
      A & 0 \\
      -C & 0
  \end{bmatrix}
  \quad
  B^* = \begin{bmatrix}
      B \\
      0
  \end{bmatrix}
  \]
- Use Ackermann’s formula or any similar method to obtain the augmented vector \(K^*\), using as input \(A^*, B^*\) and the desired \(n+1\) poles of the system.
- The resulting vector \(K^*=[K - k_*]\) will be used to completely define the new servomechanism closed loop system.

\[
\begin{bmatrix}
    A - BK & Bk_* \\
    -C & 0
\end{bmatrix}
\begin{bmatrix}
    0 \\
    1
\end{bmatrix} \quad \begin{bmatrix}
    C & 0
\end{bmatrix}
\]
Exercise 4.4: Design an integral servomechanism for the state equation of the previous example so that the closed-loop unit step response reaches a steady-state value of 1. Compare the performance of the integral action servo with the input gain method when the original system dynamics matrix is perturbed yielding the new dynamic system $\tilde{A}$.

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -14 & -6 & -2 \end{bmatrix}$$

Resulting step response: